Homework 16: Determinants

This homework is due on Friday, March 9, respectively on Tuesday, March 20, 2018.

1 Find the determinants of A, B, C:

$$A = \begin{bmatrix} 0 & 5 & 7 & 3 & 7 & 1 \\ 6 & 0 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 3 & 3 & 0 & 0 & 6 & 0 \\ 4 & 2 & 0 & 4 & 0 & 0 \\ 5 & 3 & 2 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 & 4 & 0 \\ 7 & 0 & 5 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 2 & 2 & 5 \\ 0 & 0 & 0 & 3 & 1 & 1 & 2 \\ 0 & 0 & 4 & 7 & 3 & 4 & 7 \\ 0 & 5 & 9 & 6 & 4 & 8 & 2 \\ 6 & 8 & 6 & 8 & 0 & 9 & 1 \end{bmatrix}$$

Solution:
a) There is only one pattern (Hunt it down) $\det(A) = 1515$.
b) Use Laplace expansion With the right column. This gives us $\det(B) = -3528$.
c) The only contributor to the determinant is the anti-diagonal. The number of up-crossings is $6 + 5 + 4 + 3 + 2 + 1 = 21$ which is odd. The result is $-3(1)(2)(3)(4)(5)(6) = -2160$.

2 Without doing much computation, determine whether the following determinant is positive, zero or negative:

$$\begin{bmatrix} 21 & 20^9 & 7 & -6 & 3 & 9 \\ 20^9 & 3 & 2 & 2 & 2 & 2 \\ 6 & 4 & 91 & 1 & 20^9 & -1 \\ 2 & 2 & 20^9 & 1 & -5 & 9 \\ 9 & 1 & -1 & 20^9 & 2 & 2 \\ 7 & 4 & -1 & 2 & 4 & 20^9 \end{bmatrix}.$$

Solution:
There is one dominant pattern with 3 upcrossings. The value of that pattern is 20^{9*6} There are only 719 other patterns contributing maximally 10^6 each. This is smaller than the dominant one The sign is negative.

3 a) Use the Leibniz definition of determinants to show that the **partitioned matrix** satisfies $\det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \det(A)\det(B)$.
b) Assume now that A, B are $n \times n$ matrices. Can you find a formula for $\det \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$? (It will depend on n.)
c) Show that number of up-crossings of a pattern is the same if the pattern is transposed and that therefore $\det(A^T) = \det(A)$.

Solution:
a) The determinant is $\det(A)\det(B)$.
b) The determinant is $(-1)^n \det(A)\det(B)$.
c) The total number of up-crossings minus the total number of down-crossings is also zero, meaning that they must be the same. Taking the transpose swaps up-crossings and down-crossings, and therefore the total number of up crossings for the transpose of any pattern in A is the same as that of the original pattern. We may conclude that $\det(A^T) = \det(A)$.

4 Find the determinant of the matrix $A_{ij} = 2^{ij}$ for $i, j \leq 4$. It is $\begin{bmatrix} 2 & 4 & 8 & 16 \\ 4 & 16 & 64 & 256 \\ 8 & 64 & 512 & 4096 \\ 16 & 256 & 4096 & 65536 \end{bmatrix}$. First scale some rows to make the computa-

tion more managable.

Solution:
$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 8 & 64 & 512 \\ 1 & 16 & 256 & 4096 \end{bmatrix}$ is 64512. The result is $(2*4*8*16)64512$

5 Use Laplace expansion to find a formula for the determinant of the 5×5 matrix $L(5)$ which has 2 in the diagonal and 1 in the side diagonals and 0 everywhere else. Find first the determinants of $L(2), L(3), L(4)$ and then $L(5) = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. Can you see how it continues? Optional: If you dare, try to prove your conjecture.

Solution:
We have $\det(L(2)) = 3, \det(L(3)) = 4, \det(L(4)) = 5, det(L(5)) = 6$. We suspect that $det(L(n)) = n+1$. We know that the base case $n = 1$ satisfies this. Now, if it is true for 1 through $n-1$, then using Laplace expansion, we have $det(L(n)) = 2det(L(n-1)) - det(L(n-2)$, which yields $2(n+1) - 2n = n+1$, as predicted.

Determinants I

The **determinant** of a $n \times n$ matrix A is defined as the sum $\Sigma_\pi (-1)^{|\pi|} A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)}$, where π is a permutation of $\{1, 2, \ldots, n\}$ and $|\pi|$ is the number of up-crossings in the pattern given by π. This is the Leibniz definition of determinants. By grouping the patterns according of the position in the first column, we get immediately the Laplace expansion $\det(A) = (-1)^{1+1} A_{11} \det(B_{i1}) + \cdots + (-1)^{1+n} A_{n1} \det(B_{n1})$, where for each entry a_{j1} in the first column form the $(n-1) \times (n-1)$ matrix B_{j1} in which the first column and the j'th row of A are deleted.