Homework 14: Orthogonal transformations

This homework is due on Monday, March 5, respectively on Tuesday, March 6, 2018.

1 Determine from each of the following matrices whether they are orthogonal:
a) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. d) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,
e) $\begin{bmatrix} \cos(7) & -\sin(7) & 0 & 0 \\ \sin(7) & \cos(7) & 0 & 0 \\ 0 & 0 & \cos(1) & \sin(1) \\ 0 & 0 & \sin(1) & -\cos(1) \end{bmatrix}$, f) $[-1]$, g) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

2 If A, B are orthogonal, then
a) Is A^T orthogonal? b) Is B^{-1} orthogonal? c) Is $A - B$ orthogonal? d) Is $A/2$ orthogonal? e) Is $B^{-1}AB$ orthogonal? f) Is BAB^T orthogonal?

3 a) Matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ can be multiplied and the result is of the same form. These rotation dilation matrices are also called "complex numbers"! Which of these matrices plays the role of $i = \sqrt{-1}$, that is, which of them has the property that $A^2 = -1$ (where -1 means $-I_2$)?
b) Figure out the formula for the multiplication $(a+ib)(c+id)$ of complex numbers by looking at the product $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$.
c) If you draw complex numbers $a+ib$, $c+id$ as vectors, what is the multiplication geometrically?

4 Mathematicians for a long time looked for higher dimensional analogues of the complex numbers. Matrices of the form $A(p,q,r,s) = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$ are called **quaternions**.

a) Find a basis for the set of all the matrices above.

b) Check that if $p^2+q^2+r^2+s^2 = 1$, then we have an orthogonal matrix $A(p,q,r,s)$.

5 a) Explain why the identity matrix is the only $n \times n$ matrix that is orthogonal, upper triangular and has positive entries on the diagonal. b) Conclude, using a) that the QR factorization of an invertible $n \times n$ matrix A is unique. That is, if $A = Q_1R_1$ and $A = Q_2R_2$ are two factorizations, argue why $Q_1 = Q_2$ and $R_1 = R_2$.

Orthogonal transformations

The transpose $A^T_{ij} = A_{ji}$ operation satisfies the rules $(AB)^T = B^TA^T$ and $(A^T)^T = A$. The rank of the transpose is the same as the rank of A. An $n \times n$ matrix A is **orthogonal** if $A^TA = 1 = 1_n$. The linear transformation of an orthogonal matrix is called an **orthogonal transformation**. It preserves length and angle. The column vectors of an orthogonal matrix forms an orthonormal basis. The product of two orthogonal matrices is orthogonal. The inverse A^{-1} is orthogonal and given by A^T. Examples of orthogonal transformations are rotations or reflections or the identity matrix.