Homework 14: Orthogonal transformations

This homework is due on Monday, March 5, respectively on Tuesday, March 6, 2018.

1 Determine from each of the following matrices whether they are orthogonal:

a) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. d) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

e) $\begin{bmatrix} \cos(7) & -\sin(7) & 0 & 0 \\ \sin(7) & \cos(7) & 0 & 0 \\ 0 & 0 & \cos(1) & \sin(1) \\ 0 & 0 & \sin(1) & -\cos(1) \end{bmatrix}$, f) $[-1]$, g) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution:
In each case, check that all column vectors have length 1 and are perpendicular to each other.
a) no, b) no, c) no, d) yes, e) yes, f) yes g) no

2 If A, B are orthogonal, then
a) Is A^T orthogonal? b) Is B^{-1} orthogonal? c) Is $A - B$ orthogonal? d) Is $A/2$ orthogonal? e) Is $B^{-1}AB$ orthogonal? f) Is BAB^T orthogonal?

Solution:

We either use the definition $A^T A = 1$ or the equivalent property that A preserves length.

a) Yes; $(A^T)^T = A$, and since $A^T = A^{-1}$, $(A^T)^T A^T = AA^T = AA^{-1} = 1$.

b) Yes; we have $B^{-1} = B^T$, and by c), this is orthogonal.

c) No; take $A = B = 1$.

d) No; take $A = 1$.

e) Yes; it is a composition of orthogonal transformations, each of which preserves length.

f) Yes, B^T is B^{-1} so that also here, like in e), the two vectors are orthogonal.

3 a) Matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ can be multiplied and the result is of the same form. These rotation dilation matrices are also called "complex numbers"! Which of these matrices plays the role of $i = \sqrt{-1}$, that is, which of them has the property that $A^2 = -1$ (where -1 means $-I_2$)?

b) Figure out the formula for the multiplication $(a+ib)(c+id)$ of complex numbers by looking at the product $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$.

c) If you draw complex numbers $a + ib$, $c + id$ as vectors, what is the multiplication geometrically?

Solution:

a) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. b) The product is

$$\begin{bmatrix} ac - bd & bc + ad \\ -bc - ad & ac - bd \end{bmatrix}$$

we see that $(a + ib) * (c + id) = ac - bd + i(bc + ad)$.

c) The resulting complex number has length equal to the product of their lengths, and angle equal to the sum of their angles.

4 Mathematicians for a long time looked for higher dimensional analogues of the complex numbers. Matrices of the form $A(p, q, r, s) = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$ are called **quaternions**.

a) Find a basis for the set of all the matrices above.

b) Check that if $p^2 + q^2 + r^2 + s^2 = 1$, then we have an orthogonal matrix $A(p, q, r, s)$.

Solution:

a) Put $p = 1$ and $q, r, s = 0$ to get the first basis vector which is the identity matrix. Do similarly with the other matrices. b) We have $p^1 + q^2 + r^2 + s^2 = 1$. Then, $A^T A = \begin{bmatrix} p & q & r & s \\ -q & p & -s & r \\ -r & s & p & -q \\ -s & -r & q & p \end{bmatrix} \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix} = 1$, as desired.

5 a) Explain why the identity matrix is the only $n \times n$ matrix that

is orthogonal, upper triangular and has positive entries on the diagonal. b) Conclude, using a) that the QR factorization of an invertible $n \times n$ matrix A is unique. That is, if $A = Q_1 R_1$ and $A = Q_2 R_2$ are two factorizations, argue why $Q_1 = Q_2$ and $R_1 = R_2$.

Solution:
a) First of all, since an upper triangular matrix R has R^{-1} also upper triangular while R^T will be lower triangular, we must have R diagonal, as $R^T = R^{-1}$. Since the columns have length 1 and every column has only one nonzero entry, R must be a diagonal matrix with all entries ± 1. Requiring that the entries are all positive restricts us to the identity matrix.
b) Write $Q_1 R_1 = Q_2 R_2$, then put all Q's on one side and all R's on the other. You get $Q_2^{-1} Q_1 = R_2 R_1^{-1}$, so the right hand side is upper triangular, the left is orthogonal. By a), this has to be the identity. So $Q_2 = Q_1$ and $R_2 = R_1$.

Orthogonal transformations

The transpose $A^T_{ij} = A_{ji}$ operation satisfies the rules $(AB)^T = B^T A^T$ and $(A^T)^T = A$. The rank of the transpose is the same as the rank of A. An $n \times n$ matrix A is **orthogonal** if $A^T A = 1 = 1_n$. The linear transformation of an orthogonal matrix is called an **orthogonal transformation**. It preserves length and angle. The column vectors of an orthogonal matrix forms an orthonormal basis. The product of two orthogonal matrices is orthogonal. The inverse A^{-1} is orthogonal and given by A^T. Examples of orthogonal transformations are rotations or reflections or the identity matrix.