Homework 13: Gram Schmidt and QR

This homework is due on Friday, March 2, respectively on Tuesday, March 6, 2018.

1 Perform the Gram-Schmidt process on the two vectors $\{\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}\}$.

Solution:
We begin by normalizing the first vector to get $u_1 = [1/2, 1/2, 1/2, 1/2]^T$. Next, $v_1 \cdot \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T = 5/2$, $(5/2)v_1$ from the second vector v_2 to get $w_2 = v_2 - \frac{5}{2}v_1 = \begin{bmatrix} -1 & -1 & 3 & -1 \end{bmatrix}^T /4$ Now we normalize to get $u_2 = \begin{bmatrix} -1, -1, 3, -1 \end{bmatrix}^T /(2\sqrt{3})$.

2 Find an orthonormal basis of the hyper plane $x_1 + x_2 + x_3 + x_4 + x^5 = 0$ in $\mathbf{R}^5$.

Solution:

We first find a basis by finding the kernel of $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$. One natural choice for this is

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We can now use Gram-Schmidt to make this orthonormal. First, we normalize the first vector, obtaining $v_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$. Next, we take the dot product of v_1 with the second vector, and subtract that multiple of v_1 from it to get a vector orthogonal to v_1, then normalize it to get v_2. Repeating the procedure to make the third vector orthogonal to both v_1 and v_2, followed by scaling, gives us our third basis vector v_3. Do the same with the last one. This gives us the following basis:

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

3 First find an orthonormal basis of the kernel of

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 \end{bmatrix}.$$

Then find an orthonormal basis for the image.

Solution:

First find a basis for the kernel of A, then make it orthogonal. This is done by row reduction.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \end{bmatrix}.$$

We see that there are 3 free variables. The kernel is spanned by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

We can now make this orthonormal by Gramm-Schmidt. Luckily, the first vector is already orthogonal to the latter two, so this greatly simplifies the process. We obtain $v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. The dot product of v_1 with the third vector is 0 and the dot product of v_2 with the third vector is $\frac{1}{2}$, so we get $v_3 = \sqrt{\frac{2}{3}} \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$. b) Since the first two columns have leading 1, the first two columns of A produce a basis for the image.

4 Find the QR factorization of the following three matrices

$$A = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 7 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, C = \begin{bmatrix} 12 & 5 \\ -5 & 12 \end{bmatrix}.$$

Solution:

a) We begin with applying Gram-Schmidt on the columns, but notice that luckily for us, the columns are already orthogonal to each other. Thus, only a rescaling is required to obtain Q. The diagonal matrix containing the rescalings will clearly be our R matrix.

$$Q = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

b) Because B is a vector, normalizing it will give us a unit vector, which is Q in this case. Its norm will be the R matrix.

$$Q = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} / \sqrt{30}, R = \sqrt{30}.$$

Solution:
c) Because C is a rotation dilation matrix, it is simply a dilation times an orthogonal (rotation) matrix. Thus, we simply decompose into the rotation and the dilation parts to obtain:

$$Q = \begin{bmatrix} 12/13 & -5/13 \\ 5/13 & 12/13 \end{bmatrix}, R = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}.$$

5 Find the QR factorization of the following matrices:

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix},$$

$$B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, C = \begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}.$$

Solution:
a) The first one is already the QR decomposition.
b) The matrix can be decomposed as the identity times the diagonal matrix with entries 4 c) This is a rotation dilation. We can write it as a composition of a rotation and a dilation $\frac{1}{13}\begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix} \cdot \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$

Gram Schmidt and QR

The **Gram Schmidt orthogonalization process** produces from an arbitrary basis $\mathcal{B} = \{v_j\}$ an orthonormal basis $\{u_j\}$. This goes as follows: $\vec{w}_1 = \vec{v}_1$ and $\vec{u}_1 = \vec{w}_1/|\vec{w}_1|$. To construct u_i once you've already constructed $\vec{u}_1, \ldots, \vec{u}_{i-1}$ so that they are orthonormal, make the new vector $\vec{w}_i = \vec{v}_i - \text{proj}_{V_{i-1}}(\vec{v}_i)$, where $V_{i-1} = \text{span}(u_1, \ldots, u_{i-1})$, and then normalize $\vec{w}_i$ to to get $\vec{u}_i = \vec{w}_i/|\vec{w}_i|$. Then $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is an orthonormal basis of V and the formulas

$$\vec{v}_1 = |\vec{v}_1|\vec{u}_1 = r_{11}\vec{u}_1$$

$$\vdots$$

$$\vec{v}_i = (\vec{u}_1 \cdot \vec{v}_i)\vec{u}_1 + . + (\vec{u}_{i-1} \cdot \vec{v}_i)\vec{u}_{i-1} + |\vec{w}_i|\vec{u}_i = r_{1i}\vec{u}_1 + .. + r_{ii}\vec{u}_i$$

$$\vdots$$

$$\vec{v}_n = (\vec{u}_1 \cdot \vec{v}_n)\vec{u}_1 + . + (\vec{u}_{n-1} \cdot \vec{v}_n)\vec{u}_{n-1} + |\vec{w}_n|\vec{u}_n = r_{1n}\vec{u}_1 + .. + r_{nn}\vec{u}_n$$

can be written in matrix form as $A = QR$,

$$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & | & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix},$$

where A and Q are $m \times n$ matrices and R is an $n \times n$ matrix.