Homework 10: Coordinates

This homework is due on Wednesday, February 21, respectively on Tuesday, February 20, 2018.

1 What are the $\mathcal{B}$-coordinates of the vector $\vec{v}$ in the basis $\mathcal{B}$.

$$\vec{v} = \begin{bmatrix} 7 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \mathcal{B} = \{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \} ?$$

Solution:

We form the matrix of basis vectors $S = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Since S contains a complete basis for R^4, it is an invertible matrix, and we can find its inverse from row reducing the augmented matrix $[S|I]$. In particular, we have

$$S^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -2 \end{bmatrix}.$$

From this, we can compute $S^{-1}\vec{v} = \begin{bmatrix} 5 \\ -1 \\ -1 \\ -6 \end{bmatrix}$.

2 What is the matrix B for the transformation $A = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ in the basis $\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$.

Solution:
We begin with writing down the matrix $S = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. We can invert S by the formula for the inverse of a 2×2 matrix, so $S^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$. Applying the change of basis formula gives us $B = S^{-1}AS = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$.

3 Chose a suitable basis to solve the following two problems:
a) Find the matrix A which belongs to a reflection at the plane $3x + 3y + 6z = 0$.
b) Find the matrix A which belongs to the reflection at the line spanned by $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$.

Solution:

We can pick a basis whose first element is a normal vector to the plane, for example $\mathcal{B} = \{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\}$.

This gives us the S-matrix of $S = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. We can solve for S^{-1} to get $\begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

a) The matrix that gives us a reflection about the yz plane is $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Multiplying out $A = SBS^{-1}$, we get the matrix $\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$.

b) The matrix that gives us a reflection at the z-axis is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Our desired reflection matrix is $A = SBS^{-1}$. We could multiply this out directly, but noting that this B is the negative of the B we used in a), and we are using the same basis, our computed A will be the negative as well. Thus, $A = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

4 Find the matrix A corresponding to the orthogonal projection

onto the plane spanned by the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Solution:

The projection onto the xy-plane is given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We will use the basis $\mathcal{B} = \{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\}$, where the third vector is the cross product of the first two (so we know it is perpendicular). The transformation matrix is

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix},$$

and its inverse

$$S^{-1} = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ -8 & -2 & 4 \\ 1 & -2 & 1 \end{bmatrix}.$$

Now, we can compute

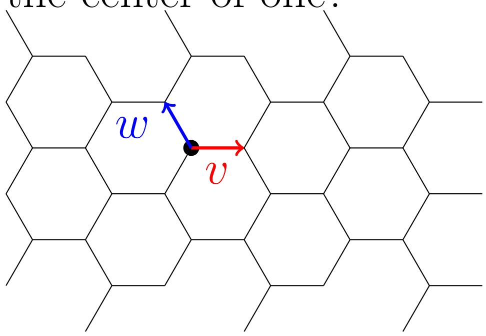
$$A = SBS^{-1} = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

5 The whole plane is covered with regular hexagons "Graphene", where the first basis vector is $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. a) Find w so that $\mathcal{B} =$

$\{v, w\}$ is the basis as seen in the picture.

b) What are the standard coordinates of the vector given in the $\mathcal{B}$ basis as $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$?

c) Is the point with $\mathcal{B}$ coordinate $\begin{bmatrix} 23 \\ 72 \end{bmatrix}$ a vertex of a hexagon or the center of one?



Picture near Harvard School of Design

Solution:

a) We know that $||\vec{w}|| = 1$ and has an angle of $120°$ with $\vec{v}$. Thus, $\vec{w} = \begin{bmatrix} \cos 120° \\ \sin 120° \end{bmatrix} = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$.

b) This vector is $2\vec{v} - \vec{w} = \begin{bmatrix} 5/2 \\ -\sqrt{3}/2 \end{bmatrix}$.

c) If we move three clicks over, we are in the same situation. So, moving 23 in the v direction is the same than moving 2 in the v direction, which is not a vertex, it is at a center. The same happens when moving in the w direction. When translating by $3w$, we are at the same point. Moving 72 in the w direction does not change anything. We are still at a center.

Coordinates

Given a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ of a linear space V, every $\vec{w}$ in V can be written as $\vec{w} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, where c_i are the **coordinates** of v. The basis defines a matrix $S = \begin{bmatrix} | \\ \vec{v}_1 \\ | \end{bmatrix}, \begin{bmatrix} | \\ \vec{v}_2 \\ | \end{bmatrix} \dots \begin{bmatrix} | \\ \vec{v}_n \\ | \end{bmatrix}$. Since $S\vec{c} = \vec{w}$ we get $\boxed{\vec{c} = S^{-1}\vec{w}}$. If A is a matrix given in the standard basis $e_1, \dots, e_n$ and B is the matrix written in the basis $\mathcal{B}$, then $\boxed{B = S^{-1}AS}$. We say B is **similar** to A. Why do we want to change basis? Because it is convenient: for example if v_1, v_2 are non-parallel vectors in a plane and v_3 is perpendicular to the plane then a projection onto the plane is the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The matrix in the standard basis is then $A = SBS^{-1}$.