Homework 9: Dimension

This homework is due on Friday, February 16, respectively on Tuesday, February 20, 2018.

1 Determine the rank and nullity of the following matrices and verify that the rank-nullity theorem holds: $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 4 & 3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution:
a) This matrix has rank is 3, the nullity is 1. It is not invertible.
b) We can row reduce A to obtain:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

There are two leading 1's, so the rank is 2. There are 2 columns with no leading 1, so the kernel has dimension 2. Also, $2+2 = 4$, the total number of columns, so the rank-nullity theorem holds.
c) This is a matrix of rank 1 and nullity 2.

2 a) In each of the 6 cases $k = 2, 3, 4, 5, 6$; give a 4×6 matrix A for which the dimension of the kernel of A is k.

b) Is there a 4×6 matrix which has a kernel of dimension 1? Explain why or why not.

Solution:

a) We proceed with our examples running from largest kernel to smallest. Note that the dimension of the kernel is going to be $6 - r$, where r is the rank. The only matrix with kernel of dimension 6 is the zero matrix $\begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$.

Any matrix with one nonzero row, ie. the matrix $\begin{bmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$ has nullity 5.

Continuing in this manner, we can construct a matrix with our desired number of non-redundant rows by picking the rows to be the first r standard basis vectors. For nullity 4, that is $r = 2$, we have $\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$.

For nullity 3, we have $\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$.

For nullity 2 , we have $\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \end{bmatrix}$.

Solution:
b) No, there is not. The kernel having dimension 1 would mean that we have only one variable. We must have at least 2 free variables however.

3 Find out whether the following set of vectors related to prime numbers forms a basis of R^4.

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 11 \\ 13 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \\ 13 \\ 17 \end{bmatrix} \right\}$$

Solution:
Let A be the matrix whose columns are these four vectors. Then, we can row reduce to get $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. By counting the leading 1s, we see that $\text{rank}(A) = 3$. This means that the column vectors span a 3 dimensional space and not all of R^4.

4 The following matrices encode the letters of "Touchy":

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, O = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$, a) Group the letters which have the same kernel.

b) Group the letters which have the same image.

Solution:

Recalling that for any matrix A, the kernel of A is equal to the kernel of rref(A), we can row reduce the matrices and compare them. So, here are the row reductions:

$$rrefT = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, rrefO = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$rrefU = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$rrefC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, rrefH = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$rrefY = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) Compute the kernel in each case. All except C have the same kernel.

b) We see the image by looking at the original Pivot columns: $imT = span(1,0,0),(1,1,1)$

$imO = span(1,1,1),(1,0,1)$

$imU = span(1,1,1),(0,0,1)$

$imC = span(1,1,1),(1,0,1)$

$imH = span(0,1,0),(1,1,1)$

$imY = span(1,0,0),(0,1,1)$

While clearly C,O have the same image, when we look also at the basis.

If we look at the linear space only, then we have the pairs TH and COH.

5 Consider the subspace V consisting of all vectors in R^4 which are

perpendicular to both $v = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

Find a basis for V.

Solution:
If u is a vector in R^4, then saying that u is perpendicular to v and w means $u \cdot v = 0$ and $u \cdot w = 0$, which gives us a system of two equations. In matrix form, this is $Au = 0$, where A is the matrix with rows v, w.
We can row reduce to get $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$. This gives us two free parameters, which we will call s and t. We can write our solution u as $[t, s, s, t] = s[0, 1, 1, 0] + t[1, 0, 0, 1]$. This gives us a basis of $\{[1, 0, 0, 1], [0, 1, 1, 0]\}$.

Basis

If V is a linear space and $\{v_1, \cdots, v_n\}$ is a basis, then n is the **dimension** of V. If $\vec{v}_1, \ldots, \vec{v}_p$ are linearly independent in V and $\vec{w}_1, \ldots, \vec{w}_q$ span V then $p \leq q$. The dimension of the image of A is called the **rank** of A. The dimension of the kernel of A is called the **nullity** of A. The rank-nullity theorem tells that the sum of the rank and the nullity is equal to the number of columns of A. It's easy to see why this is true if you remember that the rank of A is the number of leading 1's in rref(A) and the nullity of A is the number of free variables (columns without leading 1).