Homework 8: Basis

This homework is due on Wednesday, February 14, respectively on Thursday, February 15, 2018.

1 Which of the following sets are linear spaces? Check in each case the three properties characterizing a linear space. Only a brief explanation is needed (can be a picture too): a) $W = \{(x,y,z) \mid x > 0\}$
b) $W = \{(x,y,z) \mid xyz = 0\}$ c) $W = \{(x,y,z) \mid x = 2y = 3z\}$
d) $W = \{(x,y,z) \mid x = y = z + 1\}$
e) $W = \{(x,y,z) \mid x^2 + y^2 - z^2 = 0\}$ f) $W = \{(x,y,z) \mid x,y,z \text{ are rational numbers }\}$ g) $W = \{(x,y,z) \mid x = y = z = 0\}$

2 a) Write the three dimensional space $x + 2y + 3z + 4t = 0$ as a kernel of a 1×4 matrix.
b) Write the same plane as the image of a 4×3 matrix.
c) Find a basis for this space.

3 Check whether the given set of vectors is linearly independent
a) $\{\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\}$. b) $\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}\}$. c) $\{\begin{bmatrix} 3 \\ 16 \end{bmatrix}, \begin{bmatrix} 4 \\ 18 \end{bmatrix}, \begin{bmatrix} 5 \\ 19 \end{bmatrix}\}$.

4 Find a basis for the image as well as as a basis for the kernel of the following matrices

a) $\begin{bmatrix} 7 & 0 & 7 \\ 2 & 3 & 8 \\ 9 & 0 & 9 \\ 5 & 6 & 17 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

5 The orthogonal complement of a subspace V of R^n is the set $V^\perp$ of all vectors in R^n that are perpendicular to every single vector in V. Find a basis for the orthogonal complement in each case:
a) The line L in R^5 spanned by $\begin{bmatrix} 1 & 2 & 2 & 1 & 1 \end{bmatrix}^T$, (If v is a row

vector v^T denotes the corresponding column vector).

b) The plane Σ in R^4 spanned by $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^T$.

c) The space $V = \{(0,0)\}$ in the two-dimensional plane R^2.

Basis

V is a **linear space** if 0 is in V, if $v+w$ is in V for all v, w in V and if λv is in V for every v in V and every λ in $\mathbb{R}$. Examples: kernels $V = \ker(A)$ or images $V = \mathrm{im}(A)$ are linear spaces If V, W are linear spaces and V is a subset of W, then V is called a **linear subspace** of W. A line through the origin for example is a linear subspace of $\mathbb{R}^3$. A set $\mathcal{B}$ of vectors $\{v_1, \dots, v_n\}$ **spans** V if every $v \in V$ is a sum of vectors in $\mathcal{B}$. A set $\mathcal{B}$ is linear independent if $a_1 v_1 + \cdots + a_n v_n = 0$ implies $a_1 = \cdots = a_n = 0$. It is a **basis** of V if it both **spans** V and is linearly independent. Example: the standard basis vectors $\{e_1, \dots, e_n\}$ form a basis of $\mathbb{R}^n$. How do we determine whether a set of vectors is a basis of $\vec{R}^n$? Place the vectors of $\mathcal{B}$ as columns in a matrix A, then row reduce A. If every column of a matrix has a leading 1, then the set of column vectors $\mathcal{B}$ are linearly independent and the kernel of A is $\{0\}$. How do we determine whether a set of vectors is linearly independent? Place the vectors as columns of a matrix and row reduce. If there is no free variable, then we have linear independence. Example: three vectors in $\mathbb{R}^3$ are linearly independent if they are not in a common plane.