Homework 7: Image and Kernel

This homework is due on Monday, February 12, respectively on Tuesday February 13, 2018.

1 Find the kernel of the transformation $x \to Ax$, then write down a set of vectors which span the image of A.

a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 4 & 2 & 1 & 2 & 4 \\ 4 & 2 & 1 & 2 & 4 \\ 4 & 2 & 1 & 2 & 4 \\ 4 & 2 & 1 & 2 & 4 \end{bmatrix}$,

c) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$, d) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

2 a) Give an example of a transformation from R^6 to R^3 for which the image is spanned by the two vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

b) Express the kernel of the 1×4 matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ as the image of a 4×3 matrix B.

3 a) What is the image and kernel of the shear $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$?

b) What is the image and kernel of the rotation–dilation $\begin{bmatrix} 5 & 12 \\ -12 & 5 \end{bmatrix}$?

c) What is the image and kernel of the projection $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$?

d) What is the image and kernel of the reflection $\frac{1}{13}\begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$?

e) What is the image and kernel of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

4 These problems are a bit more abstract. You might want to make use of problem sessions, office hours and MQC. Let A be an arbitrary 5×5 matrix and $B = \text{rref}(A)$.
a) Is it true that $\text{im}(A) = \text{im}(B)$? Explain why or why not.
b) Is it true that $\ker(A) = \ker(B)$? Explain why or why not.
c) Be creative and find a 3×3 or 4×4 matrix for which $\text{im}(A) = \ker(A)$.

5 Let A be a $n \times n$ matrix. Is $X \subset Y$ or is $Y \subset X$?
a) $X = \text{im}(A)$ and $Y = \text{im}(A^3)$?
b) $X = \ker(A)$ and $Y = \ker(A^3)$?
c) $X = \ker(A)$ and $Y = \ker(A^3 + A^2)$?
d) $X = \text{im}(A)$ and $Y = \text{im}(A^3 + A^2)$?

Image and kernel

The **kernel** is the set of vectors x which satisfy $Ax = 0$.
The **image** of a linear map $x \to Ax$ is the set of all vectors Ax.
The columns of A **span** the image of A. Every $x \in \text{im}(A)$ can be written as a linear combination of column vectors. The image and kernel are both linear spaces: they are closed under addition, scalar multiplication and contain the zero vector. The kernel of a $n \times n$ matrix is $\{0\}$ if and only if A is invertible if and only if the image is R^n if and only if $\text{rref}(A) = 1$.