Homework 7: Image and Kernel

This homework is due on Monday, February 12, respectively on Tuesday February 13, 2018.

1 Find the kernel of the transformation $x \to Ax$, then write down a set of vectors which span the image of A.

a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 4 & 2 & 1 & 2 & 4 \\ 4 & 2 & 1 & 2 & 4 \\ 4 & 2 & 1 & 2 & 4 \\ 4 & 2 & 1 & 2 & 4 \end{bmatrix}$,

c) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$, d) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

Solution:

a) To find the kernel, we have to solve $Ax = 0$. This means we need to row reduce $[A|0]$, which is equivalent to row reducing A. We find that $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

This matrix has two leading 1 and 7 free variables $t_3, t_4, \ldots, t_8$. We can write down the solutions $x_1 = -2t_2 - 3t_3 - 4t_4 - 5t_5 + 4t_6 - 3t_7 - 2t_8 - t_9$, $x_2 = t_2, x_3 = t_3, \ldots, x_9 = x_9$. The kernel is the span of the following vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$
$\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

The image is spanned by the column vectors of A. Since all are parallel, we can write $im(A) = \text{span}(\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution:

1b) We find that $\text{rref}(A) = \begin{bmatrix} 4 & 2 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$. Therefore, the kernel is spanned by the column vectors $\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 0\ 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and and $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. The image is spanned by the column vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution:

c) We find that $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$. Therefore the kernel is spanned by the column vectors: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

The image is spanned by the column vector $\begin{bmatrix} 1 \end{bmatrix}$.

d) We find that $\text{rref}(A) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Therefore the kernel is spanned by the zero vector. The image is spanned by the column vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

2 a) Give an example of a transformation from R^6 to R^3 for which the image is spanned by the two vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

b) Express the kernel of the 1×4 matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ as the image of a 4×3 matrix B.

Solution:

a) Build a matrix which has the two vectors as columns. Perhaps the easiest example is $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$.

b) We first want find the general solution of $Ax = 0$. Luckily, A is already trivially row reduced, so this is $x_1 = -2u - 3v - 4w, x_2 = u, x_3 = v, x_4 = w$, where u, v, w are free parameters. We wish to identify this as the image of a 4×3 matrix, which is by definition a linear combination of two vectors. As written, we have the natural choice of $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ from which we form the matrix $\begin{bmatrix} -2 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3 a) What is the image and kernel of the shear $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$?

b) What is the image and kernel of the rotation–dilation $\begin{bmatrix} 5 & 12 \\ -12 & 5 \end{bmatrix}$?

c) What is the image and kernel of the projection $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$?

d) What is the image and kernel of the reflection $\frac{1}{13}\begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$?

e) What is the image and kernel of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Solution:

a) Solving $Ax = 0$, we find that we have trivial kernel. Again, this means that the image is all of R^2 because it is an invertible map.

b) Solving $Ax = 0$, we find that $x = 0$, so we have trivial kernel. We can also solve $Ax = b$ uniquely for any b, so we have R^2 as image. Equivalently, this is an invertible map.

c) Solving $Ax = 0$, we find that the kernel is spanned by $[1, -1]^T$. It is clear that the image is spanned by $[1, 1]$, as both columns are the same.

d) Solving $Ax = 0$, we find that we have trivial kernel. As in (a), this means that the image is all of R^2 because it is an invertible map.

e) Solving $Ax = 0$, we find that $\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Therefore, we have trivial kernel. The image is spanned by the columns, which we can see is also spanned by $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. We can think of this as vectors of the form $[x, y, 2y - x]$ for any x and y.

4 These problems are a bit more abstract. You might want to make use of problem sessions, office hours and MQC. Let A be an arbitrary 5×5 matrix and $B = \text{rref}(A)$.

a) Is it true that $\text{im}(A) = \text{im}(B)$? Explain why or why not.

b) Is it true that $\ker(A) = \ker(B)$? Explain why or why not.

c) Be creative and find a 3×3 or 4×4 matrix for which $\text{im}(A) = \ker(A)$.

Solution:

a) No, the image can be changed from row operations. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. The image of A is $\{(t,t)\}$ while the image of $B = \text{rref}(A)$ is $\{(t,0)\}$.

b) Yes, the kernel is preserved under each row operation (because it arises from solving $Ax = b$).

c) Note that having kernel equal to image means that we must have $A^2 = 0$, which should help us get started. Perhaps the simplest nonzero matrix with this property is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. While this is a 2×2 matrix rather than one of the desired size, we can try making a larger one that is similar. Setting $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ does the job.

5 Let A be a $n \times n$ matrix. Is $X \subset Y$ or is $Y \subset X$?

a) $X = \text{im}(A)$ and $Y = \text{im}(A^3)$?

b) $X = \ker(A)$ and $Y = \ker(A^3)$?

c) $X = \ker(A)$ and $Y = \ker(A^3 + A^2)$?

d) $X = \text{im}(A)$ and $Y = \text{im}(A^3 + A^2)$?

Solution:

a) The kernel of A is contained in the kernel of A^3 as $Ax = 0$ implies $A^3x = 0$.
b) The image of A^3 is contained in the image of A as $A^3u = A(A^2u)$.
c) The kernel of A is contained in the kernel of $A^3 + A$, as $Ax = 0$ means that $(A^3 + A)x = A^3x + Ax = 0 + 0 = 0$.
d) The image of $A^3 + A$ is contained in the image of A because $(A^3 + A)u = A(A^2u + u)$.

Image and kernel

The **kernel** is the set of vectors x which satisfy $Ax = 0$. The **image** of a linear map $x \to Ax$ is the set of all vectors Ax.
The columns of A **span** the image of A. Every $x \in \text{im}(A)$ can be written as a linear combination of column vectors. The image and kernel are both linear spaces: they are closed under addition, scalar multiplication and contain the zero vector. The kernel of a $n \times n$ matrix is $\{0\}$ if and only if A is invertible if and only if the image is R^n if and only if $\text{rref}(A) = 1$.