Homework 5: Transformations in geometry

This homework is due on Wednesday, February 7, respectively on Thursday February 8, 2018.

1 a) Find the reflection matrix at the line $y + x = 0$ in the plane.
b) Find the 2×2 rotation dilation matrix which rotates by $45°$ **counter clockwise** and scales by a factor $\sqrt{8}$.
c) Find the rotation dilation matrix which rotates around the origin by $60°$ clockwise and scales by a factor 14. d) Find the projection matrix onto the line $x - y = 0$ in the plane.

2 Name the following transformations and give a reason by stating a feature. Choose from **Dilation, Shear, Rotation, Projection, Reflection, Reflection Dilation or Rotation Dilation**.

a) $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, b) $A = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}$, c) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, d) $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$, e) $A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$, f) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, g) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

3 Match the matrices A, B, C, D, E with the transformations a)-e):
a) rotation around a line, b) orthogonal projection onto a line, c) reflection about a line, d) reflection about a plane, e) orthogonal projection onto a plane.

$$A = \frac{1}{9} \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix},$$

$$B = \frac{1}{5} \begin{bmatrix} -5 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}, \quad C = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}, \quad D = \frac{1}{5} \begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix}.$$

$$E = \frac{1}{3} \begin{bmatrix} 2 & -2 & -1 \\ -2 & -1 & -2 \\ -1 & -2 & 2 \end{bmatrix},$$

4 Assume $a^2+b^2 = 1$. One of the three transformations is a rotation, the other is a reflection about a line, the third is an orthogonal projection onto a line. Which is which? Find the inverse in the case of rotation and reflection.
a) $C = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. b) $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. c) $B = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$. d) For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is defined as $\det A = ad - bc$. Find the determinant of a shear, of a rotation, of a reflection about a line, of reflection in the origin, a projection onto the x axis, a rotation-dilation matrix with parameters (a, b).

5 The matrix multiplication introduced in the next lecture gives the entry ij of AB as the dot product of the i'th row of A with j'th column of B. Verify that the product of a reflection dilation $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ with an other reflection dilation $\begin{bmatrix} c & d \\ d & -c \end{bmatrix}$ is a rotation dilation. What is the rotation angle and what is the scaling factor? Discuss this with somebody else to make sure you understand this geometrically.

Reminders about linear transformations

A transformation T from R^m to R^n is **linear** if there is a $n \times m$ matrix A such that $T(x) = Ax$. To verify linearity, it is enough to give a matrix. If you suspect that a transformation to be not linear, check three conditions: i) $T(0) = 0$, ii) $T(\lambda x) = \lambda T(x)$ and iii) $T(x+y) = T(x)+T(y)$. To see what a transformation does, **look at the columns**. The first column is the image of the first standard basis vector, the 2. column the image of the 2. etc. In general it can be helpful to see vectors fixed. For a reflection at a line for example, any vector in the line stays
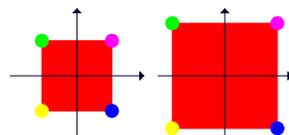
Transformations to know

Four major classes of transformations are important to know:

$$\text{rotation dilation} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{reflection dilation} \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

$$\text{shear dilation} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \quad \text{projection dilation} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

- a projection P has the property that if you apply it twice, then it is the same than applying it once. So, for any column vector v of the matrix we have $Pv = v$.

- a reflection R has the property that if you apply it twice you get back the vector. This means that if you apply the transformation to k'th column of a matrix, you should get the k'th basis vector e_k.

- if you know determinants in 2 and 3 dimensions, this can help to identify the transformation. As determinants mean signed area or signed volume. But determinants are covered later in this course.

SCALING TRANSFORMATION = DILATION

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
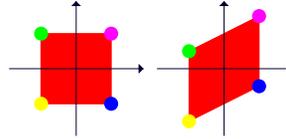
$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$
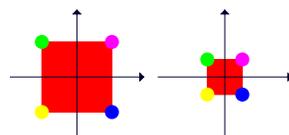
More generally, one could scale x differently then y:

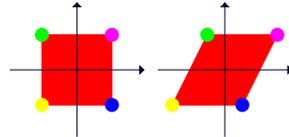$A = \begin{bmatrix} 2 & 0 \\ 0 & 1.3 \end{bmatrix}$
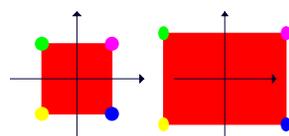
VERTICAL and HORIZONTAL SHEAR.
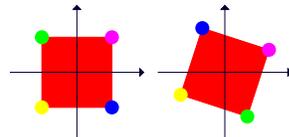
$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
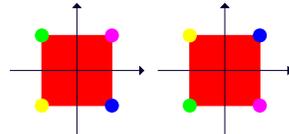
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Shears are transformation in R^2 for which there is w with $T(w) = w$ and $T(x) - x$ is a multiple of w for all x.

REFLECTION AT A LINE:

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$
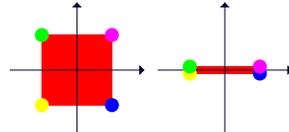
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

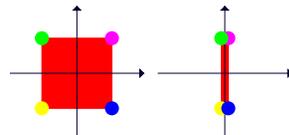The first matrix is a reflection at a unit vector with components $a = \cos(\alpha), b = \sin(\alpha)$ It an also be written as $A = \begin{bmatrix} 2a^2 - 1 & 2ab \\ 2ab & 2b^2 - 1 \end{bmatrix}$.

ORTHOGONAL PROJECTION ONTO LINE:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
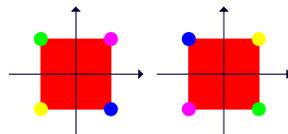
$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

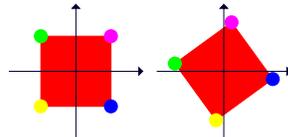If the line contains the unit vector with components a, b then $A = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$ It is already good to notice that this matrix has some symmetry when reflecting at the diagonal (ab appears twice). This symmetry is also a feature of reflection dilations but not in rotation dilations.

ROTATION AROUND A POINT:

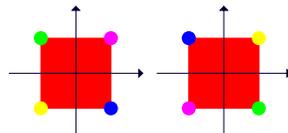$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$
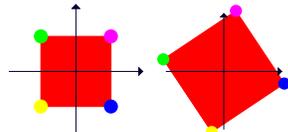
The first example $A = -1$ is reflection at the origin. Any rotation has the form of the matrix seen below.

ROTATION-DILATION:

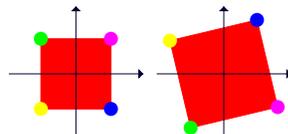$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

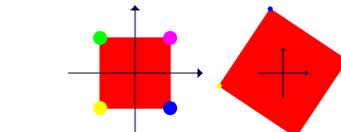$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

A rotation dilation is a composition of a rotation by angle $\arctan(y/x)$ and a dilation by a factor $\sqrt{x^2 + y^2}$. If $z = x+iy$ and $w = a + ib$ and $T(x, y) = (X, Y)$, then $X + iY = zw$. So a rotation dilation is related to the process of the multiplication with the complex number $a + ib$.

REFLECTION-DILATION:

$$A = r \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

The first example is a reflection dilation at a line with unit vector $(\cos(\alpha), \sin(\alpha)) = (a, b)$. The scaling factor is r.

Become friendly with these transformations. Why? Because knowing them helps you to navigate the field of linear algebra. Some transformations will be covered in detail later like projections or then orthogonal transformations.