Homework 4: Linear transformations

This homework is due on Monday, February 5, respectively on Tuesday February 6, 2018.

1 Which of the following transformations are linear? If it is, find the matrix A which implements the transformation.

a) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y+x \\ 0 \\ x^2-y \end{bmatrix}$ b) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7^2y \\ -3x \\ x \end{bmatrix}$ c) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+2y \\ y+z \\ 0 \end{bmatrix}$

d) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^y \\ x \end{bmatrix}$ e) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z+x \\ -3y \end{bmatrix}$ f) $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution:

a) This is not linear. The linearity condition fails. For example $T(3(x,y,z)) \neq T((x,y,z))$.

b) This is linear. The 7^2 is a constant.

c) This is linear. It is implemented by a 3×3 matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d) This is not linear. The exponential term and quadratic term means that it will not scale linearly with inputs.

e) This is linear. It is clear from the domain and range that A is a 2×3 matrix. Looking at the coefficients tells us that $A = \begin{bmatrix} -1 & -3 & 1 \end{bmatrix}$.

f) This is linear.

2 Find the inverse of the following linear transformations $x \mapsto Ax$ or state that it is not invertible

a) $A = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$ b) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ c) $A = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, e) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, f) $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$

g) Verify that $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} /(ad - bc)$ is the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $ad \neq bc$.

We will learn how to invert a matrix later. For now, get the inverse by solving $Ax = e_k$, rendering the k'th column of A^{-1}.

Solution:

a) The matrix is not invertible.

b) Here, we solve $A\vec{x} = e_1$, which tells us $2x = 1$; $A\vec{x} = e_2$ which tells us $-3y = 1$. We conclude that $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$

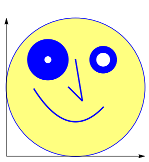
c) The matrix is not invertible.

d) By the definition of the inverse, we solve $A\vec{x} = e_1$, $A\vec{x} = e_2$ to get the first and second column, respectively. We then see that $A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$

e) This transformation swaps the two entries of a vector, so applying it twice to any vector gives us back the original vector. Therefore, the transformation is its own inverse, so $A^{-1} = A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

f) A^{-1} does not exist- we can see that the second row is -2 times the first row, so A has rank 1.

g) To check this, we need only verify that applying A to the first column gives us e_1, and applying it to the second column gives us e_2. Doing this, we get $\begin{bmatrix} \frac{1}{ad-bc}(da-bc) \\ \frac{1}{ad-bc}(ca-ac) \end{bmatrix} = e_1$ for the first column, and $\begin{bmatrix} \frac{1}{ad-bc}(db-bd) \\ \frac{1}{ad-bc}(-cb+ad) \end{bmatrix} = e_2$ for the second column.



3 For each of the matrices, sketch the effect of the linear transformation $T(x) = Ax$ on the face.

a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ e) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ f) $\begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

a) This is a shear along the y axes.

b) This is a rotation by 90° clockwise-clockwise composed with a scaling by a factor 2.

c) This is a dilation by a factor of 3.

d) This reflects at the x axes.

e) This swaps the two coordinates, so it can be thought of as a reflection along the line $x = y$.

f) This scales in the y direction by a factor 2 and in the x direction by a factor 1/3.

4 a) Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Which matrix A implements the transformation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to v \times x = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} .$$

b) Which matrix A implements the transformation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to v \cdot x = [v_1x_1 + v_2x_2 + v_3x_3] \ ?$$

c) Is there a matrix which implements the transformation $(x, y) \to (x + 1, y + 2)$? If yes, write it down.

Solution:

a) This is the cross product you recognize if you have taken multivariable calculus. Look at the image of the basis vectors e_1, e_2, and e_3 (or i, j, k as used in 21a if you have taken that course).

Appending together the output of $v \times e_1, v \times e_2$, and $v \times e_3$ as the columns, we have

$$A = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} .$$

b) This transformation is the dot product. Because the output is 1-dimensional, the representing matrix has only one row, and the columns are the values of $v \cdot e_1, v \cdot e_2$, and $v \cdot e_3$. Hence,

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} .$$

c) No, this is no linear transformation. The transformation does not map 0 to 0.

5 Find the linear transformation which reflects at the z-axes, then rotates by 180 degrees around the x-axes, then reflects at the xy-plane. Draw the images of thre three basis vectors $e_1 = [1, 0, 0]^T$, $e_2 = [0, 1, 0]^T$ and $e_3 = [0, 0, 1]^T$ to build the matrix. (To save space v^T is a column vector if v is a row vector).

Solution:
The first basis vector is mapped to $e_1 \to -e_1 \to -e_1 \to -e_1$.
The second basis vector is mapped to $e_2 \to -e_2 \to e_2 \to e_2$.
The third basis vector is mapped to $e_3 \to e_3 \to -e_3 \to e_3$.
The matrix is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is a reflection at the yz plane.

Main properties

A map T mapping x to $T(x)$ is a **linear transformation** if there is a matrix A such that $T(x) = Ax$. The transformation is invertible if x can be obtained uniquely from b. In that case the inverse is again a linear transformation.

The columns of the matrix play a key role. The image of the vector e_1 is the first column, the image of e_2 the second column etc. When solving $Ax = e_k$ we get the k'th column of the inverse.