Homework 2: Gauss-Jordan elimination

This homework is due on Wednesday, January 31, respectively on Thursday February 1, 2018.

1 Row reduce the following matrices A, B. Try to do it in as few steps as possible.

$$\text{a) } A = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}, \text{ b) } B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution:

a)

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

2 Solve the system of equations $A\vec{x} = \vec{b}$ for the Hadamard square

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 10 \\ 14 \\ 18 \end{bmatrix}$$

by row reducing the augmented 3×4 matrix $B = [A|b]$. **The Hadamard product** of two matrices is $(U * V)_{ij} = U_{ij}V_{ij}$. In this case we took $B_{ij} = i + j - 1$ and formed $A = B * B$. This is of course not the same than $B^2 = B.B$.

Solution:

The augmented Matrix is $\text{rref}(B) = \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$. The solution is $x = [0, -2, 2]^T$.

3 Find all solutions of the following system of equations using Gauss-Jordan elimination. As usual, you need to show all work.

$$\begin{aligned} 8x_1 + 6x_2 + 4x_3 - 2x_4 &= 8 \\ 5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 &= 11 \end{aligned}$$

Solution:

We don't write the division line before the last column. We have the augmented matrix

$$[A|b] = \left[\begin{array}{cccc|c} 8 & 6 & 4 & -2 & 8 \\ 5 & 4 & 3 & -1 & 4 \\ -2 & -2 & -1 & 2 & -3 \\ 11 & 6 & 4 & 1 & 11 \end{array}\right].$$

We add/subtract multiples of the first row to the remaining rows to get $\left[\begin{array}{cccc|c} 4 & 3 & 2 & -1 & 4 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 3 & 3 \end{array}\right]$. We subtract the second row from the third row and rearrange the rows to get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 2 & -1 & 4 \\ 3 & 0 & 0 & 3 & 3 \end{array}\right]$. Subtracting multiples of the first from the remaining rows allows us to clear the first column: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 3 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

Next, we subtract three times the second row from the third row to get the matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 0 & -1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. Our reduced system is $\left[\begin{array}{ccccccc} x_1 & & & + & x_4 & = & 1 \\ & x_2 & & - & 3x_4 & = & 2 \\ & & x_3 & + & 2x_4 & = & -3 \end{array}\right]$, which rearranges to $\left[\begin{array}{ll} x_1 & = 1 - x_4 \\ x_2 & = 2 + 3x_4 \\ \end{array}\right.$.

4 Two $n \times m$ matrices in reduced row-echelon form are called **of the same type** if they contain the same number of leading 1's in the same positions. For example, $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are of the same type.

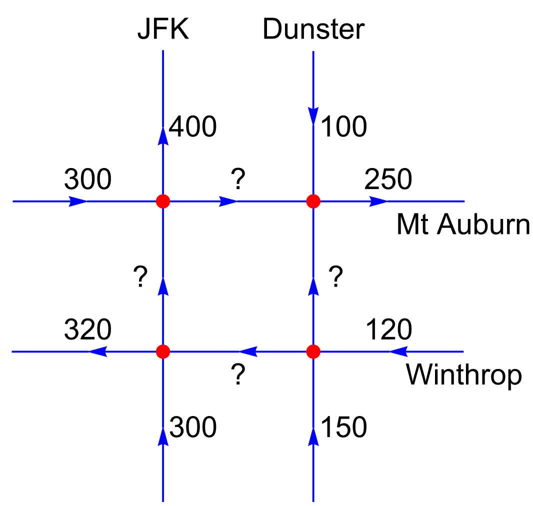
a) How many types of 2×3 matrices in reduced row-echelon form are there?

Solution:

We can have either zero, one, or two leading 1's.

In total, we have 7 types. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

5 The traffic on the streets near Harvard is indicated on the figure. Assuming that the total traffic leaving a node is the amount entering it, what can you say about the traffic at the four locations indicated by question marks? What is the highest and the lowest possible traffic volume at each location?



Solution:

Let x_1, x_2, x_3, and x_4 be the traffic volume at the four streets. We are told that the number of cars coming into each intersection is the same as the number of cars coming out:
$\begin{bmatrix} x_1 + 300 & = 320 + x_2 \\ x_2 + 300 & = 400 + x_3 \\ x_3 + x_4 + 100 & = 250 \\ 150 + 120 & = x_1 + x_4 \end{bmatrix}$. Turn over

Solution:

We can rewrite this as $\begin{bmatrix} x_1 & - & x_2 & & & & & = & 20 \\ & & x_2 & - & x_3 & & & = & 100 \\ & & & & x_3 & + & x_4 & = & 150 \\ x_1 & & & & & + & x_4 & = & 270 \end{bmatrix}$,

or in matrix form $\begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & 1 \\ 1 & & & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 20 \\ 100 \\ 150 \\ 270 \end{bmatrix}$.

This gives us an augmented matrix

$$[A|b] = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 20 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \\ 1 & 0 & 0 & 1 & 270 \end{array}\right].$$

We can begin row reduction by subtracting the first row from the fourth, giving $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 20 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \\ 0 & 1 & 0 & 1 & 250 \end{array}\right]$, followed by adding the second row to the first and subtracting it from the fourth,

Solution:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 120 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 1 & 1 & 150 \end{array}\right].$$

Finally, adding the third row to the first two rows and subtracting it from the fourth, we get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 270 \\ 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

Solution:

We see that we have one free parameter to choose, so setting $x_4 = t$, the solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 270 - t \\ 250 - t \\ 150 - t \\ t \end{bmatrix}$.

Since the x_i must be positive integers (or zero), t must be an integer with $0 \leq t \leq 150$. The lowest possible values are $x_1 = 120,\ x_2 = 100,\ x_3 = 0$, and $x_4 = 0$, while the highest possible values are $x_1 = 270,\ x_2 = 250,\ x_3 = 150$, and $x_4 = 150$.

Main definitions

The **Gauss-Jordan Elimination** process brings a matrix into **reduced row echelon form**. It consists of **elementary row operations**: **S**wap two rows. **S**cale a row. **S**ubtract a multiple of a row from an other.
The **row-reduced matrix** $\text{rref}(A)$ has three properties:
1) if a row has nonzero entries, then the first nonzero entry is **1** (**"leading 1"**).
2) if a column contains a leading 1, then the other entries in that column are 0.
3) if a row has a leading 1, then every row above has a leading 1 to the left.
The number of leading 1 in $\text{rref}(A)$ is called the **rank** of A.