Homework 1: Linear Equations

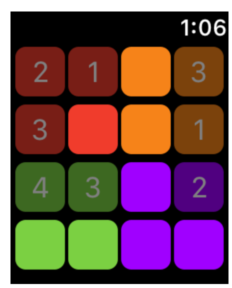
This homework is due on Monday, January 29, respectively Tuesday January 30, 2018. Homework is due at the beginning of each class in the classroom.

1 Find all solutions of the linear system

$$\left| \begin{array}{ll} x + y + z + u & = 1 \\ x + y - u + v & = 2 \\ x + z & = 3 \\ x + y + u & = 4 \\ y + v & = 5 \end{array} \right|$$

Solution:
$x = 6, y = -11, z = -3, u = 9, v = 16.$

2 On the iWatch one can play a 4×4 Suduku. The rules are that in each of the four 2×2 sub-squares, in each of the four rows and each of the four columns, the entries 1 to 4 have to appear and so add up to 10. The game initially gives 4 numbers. We have already started to enter 4 more numbers. There are still 8 missing. Introduce 8 variables and write down a system of 12 linear equations for these 8 variables, then solve the system.



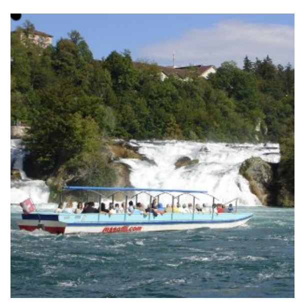
Solution:
Introduce a variable for each of the empty fields. Now write down the 12 equations. Writing it down in matrix form is not necessary. We have 12 equations for the 8 unknowns. It is easy to solve ad hoc. For example, the equation that the first row adds up to 10 is $2+1+x+3=10$. It gives $x=4$. Here is a suggestion for variables.

$$\left[\ \{2,1,x,3\}\ \ \{3,y,z,1\}\ \ \{4,3,a,2\}\ \ \{b,c,d,e\}\ \right].$$

Now we have 12 equations, like $3+y+z+1=10, 4+3+a+2=10, b+c+d+e=10$ for the rows, $2+3+4+b=10, 1+y+3+c=10, x+z+a+d=10, 3+1+2+e+10$ for the columns and $2+1+3+y=10, x+3+z+1=10, 4+3+b+c=10, a+2+d+e=10$ for the four squares The solution is

$$\left[\ \{2,1,4,3\}\ \ \{3,4,2,1\}\ \ \{4,3,1,2\}\ \ \{1,2,3,4\}\ \right].$$

3 A 10 km trip from the Swiss waterfall "Rheinfall" to the village "Rheinau" takes 30 minutes. The return trip takes an hour. How fast is the speed v (in km/h) of the boat traveling relative to the water, and how fast is the speed s (in km/h) of the river?



Solution:

Let v be the speed of the boat relative to the water, and s be the speed of the stream; then the speed of the boat relative to the land is $v+s$ downstream and $v-s$ upstream. Using the fact that (distance) = (speed)(time), we obtain the system

$$\begin{bmatrix} 10 & = (v+s)\frac{1}{2} \\ 10 & = (v-s) \end{bmatrix} \begin{matrix} \leftarrow \text{ downstream} \\ \leftarrow \text{ upstream} \end{matrix}$$

The augmented matrix is

$$\left(\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 10 \\ 1 & -1 & 10 \end{array}\right).$$

We first scale this to get

$$\left(\begin{array}{cc|c} 1 & 1 & 20 \\ 1 & -1 & 10 \end{array}\right),$$

then swap rows and eliminate to get

$$\left(\begin{array}{cc|c} 2 & 0 & 30 \\ 1 & 1 & 20 \end{array}\right).$$

Scaling gives us

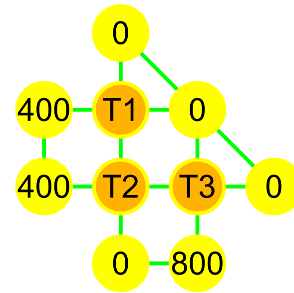
$$\left(\begin{array}{cc|c} 1 & 0 & 15 \\ 1 & 1 & 20 \end{array}\right),$$

and subtraction yields

$$\left(\begin{array}{cc|c} 1 & 0 & 15 \\ 0 & 1 & 5 \end{array}\right).$$

From this, we can see that the solution is $v=15$ and $s=5$.

4 On a heating mesh, the temperature at exterior mesh points is $0, 400$ or 800 F as given in the picture. In thermal equilibrium, each interior mesh point has the average of the temperatures at the 4 adjacent points. For example $T_2 = (T_3 + T_1 + 400 + 0)/4$. Find the temperatures T_1, T_2, T_3.



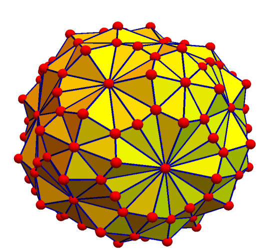
Solution:
The thermal equilibrium condition requires that

$$T_1 = \frac{T_2 + 400 + 0 + 0}{4},$$
$$T_2 = \frac{T_1 + T_3 + 400 + 0}{4},$$
$$T_3 = \frac{T_2 + 800 + 0 + 0}{4}.$$

We can rewrite this system as $\begin{bmatrix} -4T_1 + T_2 & = -400 \\ T_1 - 4T_2 + T_3 & = -400 \\ T_2 - 4T_3 & = -800 \end{bmatrix}$.

The solutions are $T_1 = 150, T_2 = 200, T_3 = 250$.

5 A polyhedron has v vertices, e edges and f triangular faces. Euler proved his famous formula $v - e + f = 2$ for such "discrete spheres". There is an other relation, $3f = 2e$ called a Dehn-Sommerville relation which always holds. Lets call this number f the "area". You get a polyhedron with area f=288. Write down a system of equations in matrix form $Ax = b$. Then determine the number of vertices and edges.



Solution:
We already know that the number of faces is 288, as each face has unit area, so there is the relation $f = 288$. Using this, along with the two given relations, we obtain the system with augmented matrix $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & 288 \end{array}\right)$. We can very easily eliminate the values in the left column to obtain $\left(\begin{array}{ccc|c} 1 & -1 & 0 & -286 \\ 0 & 2 & 0 & 864 \\ 0 & 0 & 1 & 288 \end{array}\right)$. From this, we can scale the middle row and add it to the top row, getting $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 146 \\ 0 & 1 & 0 & 432 \\ 0 & 0 & 1 & 288 \end{array}\right)$. This tells us that there are 146 vertices, and 432 edges and 288 faces.
What you can see in this problem is the general fact that for a polyhedron with triangular faces, the volume alone determines all the combinatorics!

Main definitions

A **linear equation** for variables $x_1, x_2, \ldots, x_n$ is an equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. Given m equations of this type, we get a **system of linear equations**. It can be written in matrix form $A\vec{x} = \vec{b}$, where $\vec{x}$ is a column vector containing the n variables and the $m \times n$ matrix A lists the $m \cdot n$ coefficients and $\vec{b}$ is the column vector. The system $x + 2y + z = 8, 3x - y - 7z = 4$ for example can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} .$$