

Homework 29: Fourier I

This homework is due on Wednesday, April 19, respectively on Thursday, April 20, 2017.

- 1 a) Find the angle between the functions $f(x) = x$ and $g(x) = x^6$.
 b) Project $f(x) = \sin^2(x)$ onto the plane spanned by $\sin(2x)$, $\cos(2x)$.
 c) Find the length of the function $f(x) = x^2$ in C_{per}^∞ .
- 2 Verify that the functions $\cos(nx)$, $\sin(nx)$, $1/\sqrt{2}$ form an orthonormal family.
- 3 Find the Fourier series of the function $f(x) = 7 - |3x|$.
- 4 Find the Fourier series of the function $4 \cos(3x) + \sin^2(13x) + 10$.
- 5 Find the Fourier series of the function $f(x) = |\sin(x)|$.

Fourier Series I

We write 2π periodic functions always as functions on $[-\pi, \pi]$. In C_{per}^∞ we have the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$. This defines the **length** $|f| = \sqrt{\langle f, f \rangle}$ and angle $\cos(\alpha) = \langle f, g \rangle / (|f||g|)$. The functions $\sin(nx)$, $\cos(nx)$, $1/\sqrt{2}$ form an orthonormal basis in C_{per}^∞ in the sense that they are linearly independent and span the space. Any function f can be written uniquely as a linear combination of these terms. With the Fourier coefficients $a_0 = \langle f, 1/\sqrt{2} \rangle$, $a_n = \langle f, \cos(nx) \rangle$ and $b_n = \langle f, \sin(nx) \rangle$, we can write

$$f = a_0/\sqrt{2} + \sum_n a_n \cos(nx) + \sum_n b_n \sin(nx)$$

This is called the Fourier series of f .

Example: Find the length of the function $f(x) = x^2$.

Solution: since

$$\|f\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = 2\pi^4/5 ,$$

we have $\|f\| = \sqrt{2/5}\pi^2$.

Example: find the Fourier series of the function $f(x) = x^2$ on $[-\pi, \pi]$.

Solution: the function is an even function. So, all the coefficients b_n are zero. We compute $a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \sqrt{2}\pi^2/3$ and $a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx = 4 \cos(n\pi)/n^2 = 4(-1)^n/n^2$. The Fourier series is

$$a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \pi^2/3 + \sum_{n=1}^{\infty} 4 \cos(n\pi)/n^2 \cos(nx) .$$

Below you see the sum with 3 terms. It already gives quite a good fit of the parabola which is continued to a periodic function.

