

## Homework 27: Differential operators

This homework is due on Friday, April 14, respectively on Tuesday, April 18, 2017.

The linear spaces  $C^\infty$ ,  $C_{\text{per}}^\infty$ ,  $P$  and  $T$  are defined on the next page.

- 1 The linear map  $Df(x) = f'(x)$  is an example of a **differential operator**. It has the constant functions as the kernel. This means that there is no unique inverse. One inverse is  $Sf(x) = D^{-1}f(x) = \int_0^x f(t) dt$ .
  - a) Evaluate  $D \sin$ ,  $D \cos$ ,  $D \tan$ ,  $S1/(1+x^2)$ ,  $S \tan$ .
  - b) Can you find an eigenfunction (= eigenvector)  $f$  of  $D$  to the eigenvalue 33?
  - c) Verify that if  $f$  is an eigenfunction of  $D$  to the eigenvalue 4, then  $f$  is also an eigenfunction of  $D^5 + 3D + 7$ . What is the eigenvalue?
  
- 2 a) Find a solution of the equation  $D^2f = 2x + 1/x$  on the space  $C^\infty((0, \infty))$  of all smooth functions on the positive real axes.
  - b) Find two linearly independent solutions of the eigenvalue equation  $D^2f = -9f$  on the space  $C_{\text{per}}^\infty$
  
- 3 a) Find a basis for the kernel of  $D^4$  on the linear space  $P$  of polynomials.
  - b) Find the image  $D^3 + D + 1$  on the linear space  $P$ ?
  - c) Find the eigenvalues of  $D^3 + D + 1$  on the space  $C_{\text{per}}^\infty$  of smooth periodic functions with period  $2\pi$ .
  - d) Find the kernel of  $Af = (D - \sin(t))f(t)$  on  $C_{\text{per}}^\infty$ .
  
- 4 a) Check that  $Qf(x) = xf(x)$  and  $Pf(x) = iDf(x)$  satisfy the Heisenberg commutation relation:  $(PQ - QP)f = if$ .
  - b) Check that for any real  $\omega$ , the function  $e^{i\omega t}$  is an eigenfunction of  $iD$  in  $C^\infty$ .
  - c) Check that on  $C_{\text{per}}^\infty$ , only the functions  $e^{i\omega t}$  with integer  $\omega$  are eigenfunctions. (Momentum  $\omega$  is quantized.)

- 5 a) Verify that  $Sf(x) = \int_0^x f(t) dt$  is a linear operator on the linear space  $C^\infty$  of smooth functions.  
 b) Show that  $DSf(x) = f(x)$  and c) show that  $SDf(x) = f(x) - f(0)$ . What is the theorem?

## Differential operators

A function is **smooth** if it can be differentiated arbitrarily often. The space  $C^\infty$  of real valued **smooth functions** is a linear space: if  $f, g$  are in  $C^\infty$ , then  $f + g$ , the zero function  $0$  is in  $C^\infty$  and  $\lambda f$  is in  $C^\infty$  for every real  $\lambda$ .  $C^\infty$  contains the linear space  $P$  of all **polynomials**. The space  $C_{\text{per}}^\infty$  of smooth periodic functions with period  $2\pi$  forms a linear space too. It contains the linear subspace  $T$  of **trigonometric polynomials**. The space  $P$  of **polynomials** is spanned by  $\{1, x, x^2, x^3, \dots\}$  and the space  $T$  of trigonometric polynomials is spanned by  $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \dots\}$ . They are infinite dimensional. The space  $P_3$  of cubic polynomials  $d + cx + bx^2 + ax^3$  is 4-dimensional as it has the basis  $\{1, x, x^2, x^3\}$ . The transformation map  $D : f \rightarrow f'$  is linear: it satisfies  $D(f + g) = Df + Dg$ ,  $D(\lambda f) = \lambda Df$  and  $D0 = 0$ . We call any polynomial of  $D$  like  $D^2 - D + 1$  a **differential operator**. The linear map  $D$  on  $C^\infty$  has as the kernel the one dimensional space of constant functions. What are the eigenvalues and eigenvectors of  $D$ ? Because  $De^{\lambda x} = \lambda e^{\lambda x}$ , every real number  $\lambda$  is an eigenvalue on  $C^\infty$ . The linear map  $D$  has no real eigenvalues on  $C_{\text{per}}^\infty$  but complex eigenvalues  $in$  as  $De^{inx} = ine^{inx}$ , where  $n$  is an integer. The fact that they are quantized is the reason why quantum mechanics is called “quantum” (the operator  $P = iD$  is called “momentum”) and  $Qf = xf$  “position”). The square  $-D^2$  has now real eigenvalues  $n^2$ , where  $n$  is an integer. It is the energy operator of a particle on the circle. The eigenfunctions are  $1, \sin(nx)$  and  $\cos(nx)$ . We are interested in  $D$  because it will allow us to solve differential equations like  $(D^2 + 5D + 6)f = \sin(5x)$ .