

Homework 25: Differential equations II

This homework is due on Monday, April 10, respectively on Tuesday, April 11, 2017.

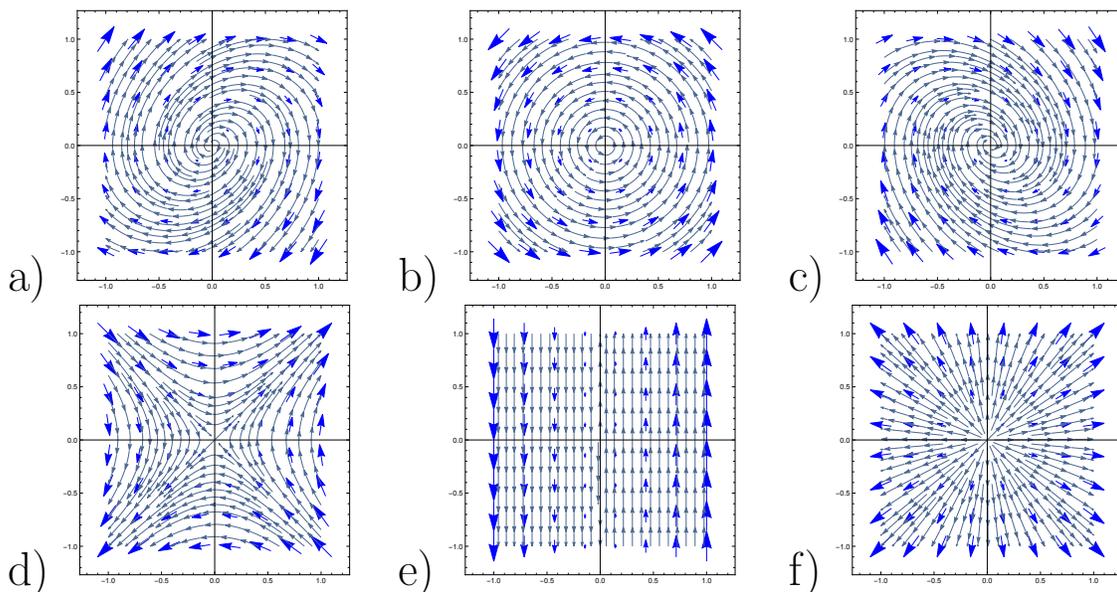
- 1 a) Find the solution to the differential equation $\frac{d^2x}{dt^2} = -x$ with initial conditions $x(0) = 5$, $\frac{dx}{dt}(0) = 0$ by writing the initial condition $\vec{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ as a linear combination of eigenvectors $[i, 1]^T$ and $[i, -1]^T$ of A , where

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

is the system written in vector form using 2×2 matrix A .

- 2 Match the differential equations $\frac{dx}{dt} = Ax$ with the phase portraits.

i) $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ ii) $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, iii) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
 iv) $A = \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}$ v) $A = \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}$ vi) $A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$



3 Determine the stability of the systems

$$a)v'(t) = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & -2 & 5 & -1 \\ -2 & -2 & 3 & 4 \\ 3 & -4 & 1 & -3 \end{bmatrix} v(t), \quad b)v'(t) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & -3 & -4 \\ 1 & -1 & 4 & -3 \end{bmatrix} v(t).$$

4 To solve the fourth order equation $w''''(t) = w(t)$, we write it as a system of first order differential equations of the form $\vec{v}'(t) = A\vec{v}(t)$ using $\vec{v}(t) = (x(t), y(t), z(t), w(t))$ where $w'(t) = z(t)$, $z'(t) = y(t)$, $y'(t) = x(t)$, $x'(t) = w(t)$ and A is a 4×4 matrix. Write down a general closed form solution formula $\vec{v}(t) = c_1 e^{\lambda_1 t} \vec{v}_1(t) + \dots + c_4 e^{\lambda_4 t} \vec{v}_4(t)$, where c_1, c_2, c_3, c_4 are parameters. Is this system stable or unstable?

5 True or false? Give short explanations

- If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = A^2x$ is stable.
- If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = A^T x$ is stable.
- If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = 2Ax$ is stable.
- If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = -Ax$ is stable.
- If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = (A - I_n)x$ is stable.

Differential Equations II

$\frac{d^2x}{dt^2} = -k^2x(t)$ is called **harmonic oscillator**. It has solutions $x(t) = a \cos(kt) + b \sin(kt)$, where a, b depend on initial conditions. It becomes a system $\frac{dx}{dt} = y(t)$, $\frac{dy}{dt} = -k^2x(t)$. In general, for a $n \times n$ matrix A , the system $v' = Av$ is **stable** if all eigenvalues satisfy $\text{Re}(\lambda_j) < 0$.