

## Homework 24: Differential equations I

This homework is due on Friday, April 7, respectively on Tuesday, April 11, 2017.

- 1 a) Solve the differential equation  $\frac{dx}{dt} = 9/x^2$ , with  $x(0) = 2$ .  
 b) Solve the differential equation  $\frac{dx}{dt} = 1 + x^2$ , with  $x(0) = 0$ .  
 c) Solve the differential equation  $\frac{dx}{dt} = 1/\cos(x)$ , with  $x(0) = 0$ .

- 2 Solve the system

$$\frac{dx}{dt} = Ax, \quad A = \begin{bmatrix} 3 & 9 \\ 7 & 5 \end{bmatrix}$$

with initial condition  $x(0) = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$ .

- 3 a) For which real  $p, q$  is the system  $\frac{dx}{dt} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} x(t)$  stable? b)

For which real  $p$  is the system  $\frac{dx}{dt} = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} x(t)$  stable?

- 4 The interaction of two animal species is modeled by the equations

$$\begin{aligned} \frac{dx}{dt} &= 1.5x - 1.2y \\ \frac{dy}{dt} &= 0.8x - 1.4y \end{aligned}$$

- a) Interpret the system. Is it a symbiosis, competition or predator-prey?  
 b) Sketch the phase portrait in the first quadrant.  
 c) What happens in the long term? Does it depend on the initial population? If so, how?

- 5 A door opens on one side only. A spring mechanism closes the door which forms an angle  $\theta(t)$  with the frame. The angular velocity is  $\omega(t) = \frac{d\theta}{dt}(t)$ . The differential equations are

$$\begin{aligned}\frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -2\theta - 3\omega\end{aligned}$$

The first equation is the definition, the second incorporates the force  $-2\theta$  of the spring and the friction  $-3\omega$ .

Sketch a phase portrait for the system and use this to answer the question, for which initial conditions, the door slams (reaches  $\theta = 0$  with negative  $\omega$ ).

## Differential Equations I

$\frac{dx}{dt} = f(x)$  is a differential equation. Solve it by separation of variables. For example, if  $\frac{dx}{dt} = t/x^2$ ,  $x(0) = 0$ , then  $x^2 dx = t dt$ . Integrate both sides to get  $t^2/2 = x^3/3 + c$  so that  $x(t) = (3(t^2/2 - c))^{1/3}$ . As  $x(0) = 0$  we have  $c = 0$  and  $x(t) = (3t^2/2)^{1/3}$ . The linear differential equation  $\frac{dx}{dt} = kx$  has the solution  $x(t) = e^{kt}x(0)$ . For  $k > 0$ , this means exponential growth. For  $k < 0$ , exponential decay. A linear system of differential equations is  $\frac{dx}{dt} = Ax$ . If  $x(0) = v$  is an eigenvector with eigenvalue  $\lambda$ , then  $x(t)$  is always a multiple of  $v$ , say  $x(t) = c(t)v$  where  $\frac{dc}{dt} = \lambda v$ . Thus if  $x(0) = c_1 v_1 + \dots + c_n v_n$  writes an initial condition as a sum of eigenvectors, then  $x(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_n e^{\lambda_n t} v_n$  is the closed form solution of the system. It is asymptotically stable, if  $x(t) \rightarrow 0$  for all initial conditions  $x(0)$ . Asymptotic stability holds if and only if  $\text{Re}(\lambda_j) < 0$  for all  $j$ .