Homework 23: Symmetric matrices

This homework is due on Monday, April 3, respectively on Tuesday, April 4, 2017.

1 Give a reason why its true or provide a counterexample.
a) The sum of two symmetric matrices is symmetric.
b) The sum of two anti-symmetric matrices is anti-symmetric.
c) The product of two symmetric matrices is symmetric.
d) The inverse of an invertible symmetric matrix is symmetric.
e) If B is an arbitrary $n \times m$ matrix, then $A = B^T B$ is symmetric.
f) If A is similar to B and A is symmetric, then B is symmetric.
g) $A = SBS^{-1}$ with $S^T S = I_n$, A symmetric $\Rightarrow B$ is symmetric.
h) Only the zero matrix is both anti-symmetric and symmetric.

2 Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2018 & 2 & 3 & 4 & 5 \\ 2 & 2021 & 6 & 8 & 10 \\ 3 & 6 & 2026 & 12 & 15 \\ 4 & 8 & 12 & 2033 & 20 \\ 5 & 10 & 15 & 20 & 2042 \end{bmatrix}.$$

3 a) Find the eigenvalues and an orthonormal eigenbasis to $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$

b) Use eigenvalues to find the determinant of $B = \begin{bmatrix} 5 & 2 & 2 & 2 & 2 \\ 2 & 5 & 2 & 2 & 2 \\ 2 & 2 & 5 & 2 & 2 \\ 2 & 2 & 2 & 5 & 2 \\ 2 & 2 & 2 & 2 & 5 \end{bmatrix}.$

4 Group matrices which are similar.

$$A=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad C=\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

5 Find the eigenvalues and eigenvectors of the Laplacian of the Bunny graph. The Laplacian of a graph with n nodes is the $n \times n$ matrix A which for $i \neq j$ has $A_{ij} = -1$ if i, j are connected and 0 if not. The diagonal entries A_{ii} are chosen so that each row add up to 0.

$$A = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$
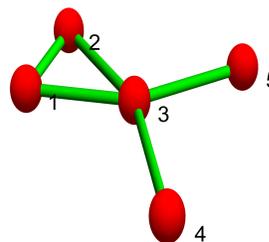
Symmetric matrices

A matrix A is **symmetric** if $A^T = A$ and **anti-symmetric** if $A^T = -A$. Projections or reflections are symmetric. Symmetric matrices appear in physics or statistics: observables like energy, position or momentum matrices are symmetric, correlation matrices are symmetric. In multi-variable calculus you have seen that the Hessian matrix consisting of the second derivatives is symmetric. The spectral theorem tells that a symmetric matrix has real eigenvalues, that it has an orthonormal eigenbasis and can be diagonalized as $S^{-1}AS = B$ with an orthogonal matrix S.