

## Homework 22: Stability

This homework is due on Friday, March 31, respectively on Tuesday, April 4, 2017.

1 Determine the stability of the dynamical system  $x(t+1) = Ax(t)$ :

a) 
$$\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}.$$

b) 
$$\begin{bmatrix} 0.9 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

2 For which constants  $a$  is the system  $x(t+1) = Ax(t)$  stable?

a) 
$$A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}.$$

b) 
$$A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}.$$
      c) 
$$A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}.$$

3 For which real values  $k$  does the drawing rule

$$x(t+1) = x(t) - ky(t)$$

$$y(t+1) = y(t) + kx(t+1)$$

produces trajectories which are ellipses?

4 Find the eigenvalues of

$$\begin{bmatrix} 0 & a & b & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & a & b \\ b & 0 & 0 & 0 & a \\ a & b & 0 & 0 & 0 \end{bmatrix}$$

Where  $a, b$  are arbitrary constants. Verify that the discrete dynamical system is stable for  $|a| + |b| < 1$ .

- 5 In the following, answer the question and give a short explanation. We say  $A$  is stable if the origin  $\vec{0}$  is a stable equilibrium.
- a) True or false: the zero matrix is stable.
  - b) True or false: the identity matrix is stable.
  - c) True or false: any reflection matrix is stable.
  - d) True or false: every horizontal shear is stable.
  - e) True or false:  $A$  is stable if and only if  $A^T$  is stable.
  - f) True or false:  $A$  is stable if and only if  $A^{-1}$  is stable.
  - g) True or false:  $A$  is stable if and only if  $A + 1$  is stable.
  - h) True or false:  $A$  is stable if and only if  $A^2$  is stable.
  - i) True or false:  $A$  is stable if  $A^2 = 0$ .
  - j) True or false:  $A$  is unstable if  $A^2 = A$ .
  - k) True or false:  $A$  is stable if  $A$  is diagonalizable.

## Stability

A discrete dynamical system  $x(t+1) = Ax(t)$  is **asymptotically stable** if  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  for all initial conditions  $x(0)$ . (If we say "stable" we always mean asymptotically stable). The main result covered in this section is that a system is asymptotically stable if and only all eigenvalues of  $A$  have absolute value  $|\lambda_j| < 1$ . For example, a rotation dilation  $A$  with first column  $Ae_1 = \begin{bmatrix} a \\ b \end{bmatrix}$  is stable if and only if  $a^2 + b^2 < 1$ . We often just say " $A$  is stable" rather than "the origin is stable for the discrete dynamical system  $x \mapsto Ax$ ".