

## Homework 15: Data fitting

This homework is due on Wednesday, March 8, respectively on Thursday, March 9, 2017.

- 1 a) Find the least square solution  $x^*$  of the system  $Ax = b$  with

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

- b) What is the matrix  $P$  which projects on the image of  $A$ ?

- 2 Find the function  $y = f(x) = ax + bx^3$ , which best fits the data

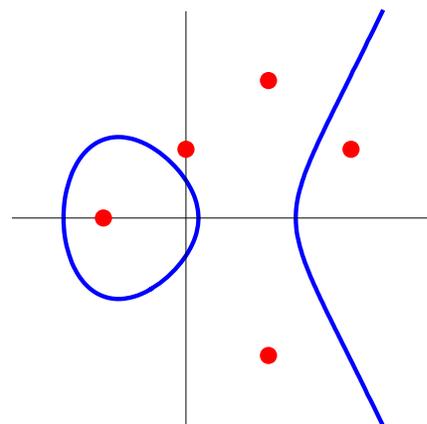
x	y
-1	4
2	3
-2	7

- 3 A curve of the form

$$y^2 = x^3 + ax + b$$

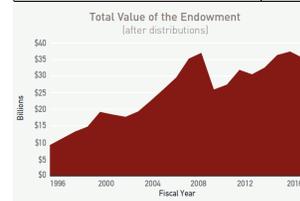
is called an **elliptic curve** in Weierstrass form. Elliptic curves are important in cryptography. Use data fitting to find the best parameters  $(a, b)$  for an elliptic curve given the following points:

$$\begin{aligned} (x_1, y_1) &= (1, 2) \\ (x_2, y_2) &= (-1, 0) \\ (x_3, y_3) &= (2, 1) \\ (x_4, y_4) &= (0, 1) \end{aligned}$$



4 A graphic from the Harvard Management Company Endowment Report of September 2016 is shown to the right. Assume we want to fit the growth using functions  $1, x, x^2$  and assume the years are numbered starting with  $1991 = 0, 1996 = 1, 2001 = 2, 2006 = 3, 2011 = 4, 2016 = 5$ . What is the best parabola  $a + bx + cx^2 = y$  which fits these data?

quintennium	billions
0	5
1	10
2	18
3	23
4	31
5	35



5 Find the circle  $a(x^2 + y^2) + b(x + y) = 1$  which best fits the data

$x$	$y$
0	1
-1	0
1	-1
1	1

In other words, find the least square solution for the system of equations for the unknowns  $a, b$  which aims to have all 4 data points  $(x_i, y_i)$  on the circle.

## Data fitting

Given a system  $Ax = b$ . Any solution of  $(A^T A)x = A^T b$  is called a **least square solution** (these always exist). (Reason: solve  $A^T(Ax - b) = 0$  for  $x$ , assuring that  $Ax - b$  is perpendicular to  $\text{im}(A)$ .) The least square solution is unique if  $A$  has a trivial kernel. In that case  $x = (A^T A)^{-1} A^T b$ . The matrix  $A(A^T A)^{-1} A^T$  is now the projection matrix onto  $\text{im}(A)$ . If the columns of  $A$  are orthonormal, this simplifies to  $P = AA^T$ .