Homework 4: Linear transformations

This homework is due on Monday, February 9, respectively on Tuesday February 10, 2017.

1 Which of the following transformations are linear? If it is, find the matrix A which implements the transformation.

a) $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 + 7y \\ -3x \\ x \end{bmatrix}$
b) $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + 2y \\ y + z \\ 0 \end{bmatrix}$

c) $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y^2 + 2x - e^x \\ x \end{bmatrix}$
d) $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z + x - 3y \end{bmatrix}$

e) $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y + x \\ 0 \\ xy \end{bmatrix}$

**2** Find the inverse of the following linear transformations $x \to Ax$ or state that it is not invertible

a) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$
b) $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$
c) $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$,
e) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,
f) $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$

g) Verify that $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / (ad - bc)$ is the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $ad \neq bc$.

We will learn how to invert a matrix later. For now, get the inverse by solving $Ax = e_k$, rendering the k'th column of A^{-1}.
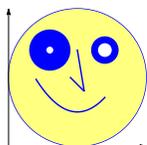
3 For each of the matrices, sketch the effect of the linear transformation $T(x) = Ax$ on the face.

a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,
b) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$,
c) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,
d) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$,
e) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

4 a) Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Which matrix A implements the transformation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to v \times x = \begin{bmatrix} v_2 x_3 - v_3 x_2 \\ v_3 x_1 - v_1 x_3 \\ v_1 x_2 - v_2 x_1 \end{bmatrix}.$$

b) Which matrix A implements the transformation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to v \cdot x = [v_1 x_1 + v_2 x_2 + v_3 x_3] \ ?$$

c) Is there a matrix which implements the transformation $x \to v + x$ if $v \neq 0$? If yes, write it down.

5 Find the linear transformation which reflects at the z-axes, then rotates by 180 degrees around the x-axes, then reflects at the xy-plane. Draw the images of $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and using these vectors to build the matrix.

Main properties

If A is a matrix then the map $x \to Ax = b$ is called a **linear transformation**. It is invertible if x can be obtained uniquely from b.

The columns of the matrix play a key role. The image of the vector e_1 is the first column, the image of e_2 the second column etc. When solving $Ax = e_k$ we get the k'th column of the inverse.