Name:

MWF 9 Fabian Haiden	
MWF 10 Ziliang Che	
MWF 10 Jeremy Hahn	
MWF 11 Rosalie Belanger-Rioux	
MWF 11 Yu-Wen Hsu	
MWF 12 Peter Garfield	
TThu 10 Oliver Knill	
TThu 11:30 Alex Perry	
TThu 11:30 Rong Zhou	

- Please fill in your name and mark your section.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or un-staple the packet.

- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) ☐ T ☐ F If A is a non-invertible $n \times n$ matrix, then $\det(A) \neq \det(\mathrm{rref}(A))$.

2) ☐ T ☐ F If the rows of a square matrix form an orthonormal basis, then the columns must also form an orthonormal basis.

3) ☐ T ☐ F A 3×3 matrix A for which the sum of the first two columns is the third column has zero determinant.

4) ☐ T ☐ F A 2×2 rotation matrix $A \neq I_2$ does not have any real eigenvalues.

5) ☐ T ☐ F If A and B both have $\vec{v}$ as an eigenvector, then $\vec{v}$ is an eigenvector of AB.

6) ☐ T ☐ F If A and B both have λ as an eigenvalue, then λ is an eigenvalue of AB.

7) ☐ T ☐ F Similar matrices have the same eigenvectors.

8) ☐ T ☐ F If a 3×3 matrix A has 3 independent eigenvectors, then A is similar to a diagonal matrix.

9) ☐ T ☐ F If a square matrix A has non-trivial kernel, then 0 is an eigenvalue of A.

10) ☐ T ☐ F If the rank of an $n \times n$ matrix A is less than n, then 0 is an eigenvalue of A.

11) ☐ T ☐ F Two diagonalizable matrices whose eigenvalues are equal must be similar.

12) ☐ T ☐ F A square matrix A is diagonalizable if and only is A^2 is diagonalizable.

13) ☐ T ☐ F If a square matrix A is diagonalizable, then $(A^T)^2$ is diagonalizable.

14) ☐ T ☐ F The matrices $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ are similar.

15) ☐ T ☐ F There exist matrices A with k distinct eigenvalues whose rank is strictly less than k.

16) ☐ T ☐ F If A is an $n \times n$ matrix which satisfies $A^k = 0$ for some positive integer k, then all the eigenvalues of A are 0.

17) ☐ T ☐ F If a 3×3 matrix A satisfies $A^2 = I_3$ and A is diagonalizable, then A must be similar to the identity matrix.

18) ☐ T ☐ F A and A^T have the same eigenvectors.

19) ☐ T ☐ F The least squares solution of a system $A\vec{x} = \vec{b}$ is unique if and only if $\ker(A) = 0$.

20) ☐ T ☐ F The matrix $\begin{bmatrix} 1 & 1000 & 1 & 1 \\ 1000 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1000 \\ 1 & 1 & 1000 & 1 \end{bmatrix}$ is invertible.

Match the following matrices with the correct label. No justifications are needed. Fill in a),b),c),d),e) into the boxes.

A) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \\ 4 & 7 & 9 \end{bmatrix}$ $\quad\square$

B) $\begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$ $\quad\square$

C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad\square$

D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad\square$

E) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad\square$

a) (2 points) skewsymmetric matrix
b) (2 points) nondiagonalizable matrix
c) (2 points) orthogonal projection
d) (2 points) symmetric matrix
e) (2 points) orthogonal matrix

Problem 3) (10 points)

Find a basis for the subspace V of $\mathbf{R}^4$ given by the equation $x + 2y + 3z + 4w = 0$. Find the matrix which gives the orthogonal projection onto this subspace.

Problem 4) (10 points)

3

Assume that A is a skew-symmetric matrix, that is, it is a $n \times n$ matrix which satisfies $A^T = -A$.

a) Find $\det(A)$ if n is odd.

b) What possible values can $\det(A)$ have if $n = 2$?

c) Verify that if λ is an eigenvalue of A, then $-\lambda$ is also an eigenvalue of A.

Problem 5) (10 points)

The recursion
$$u_{n+1} = u_n - u_{n-1} + u_{n-2}$$
is equivalent to the discrete dynamical system
$$\begin{bmatrix} u_{n+1} \\ u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \\ u_{n-2} \end{bmatrix} = A \begin{bmatrix} u_n \\ u_{n-1} \\ u_{n-2} \end{bmatrix}.$$
a) Find the (real or complex) eigenvalues of A.

b) Is there a vector $\vec{v}$ such that $||A^n \vec{v}|| \to \infty$?

c) Can you find any positive integer k such that $A^k = I_3$?

Problem 6) (10 points)

Let A be the matrix
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$
a) Find $\det(A)$.

b) Find all eigenvalues whether real or complex of A and state their algebraic multiplicities.

c) For each real eigenvalue λ of A find the eigenspace and the geometric multiplicity.

Problem 7) (10 points)

4

Find S and a diagonal matrix B such that $S^{-1}AS = B$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

Problem 8) (10 points)

Find the function of the form

$$f(t) = a\sin(t) + b\cos(t) + c$$

which best fits the data points $(0,0), (\pi, 1), (\pi/2, 2), (-\pi, 3)$.

Problem 9) (10 points)

Let V be the image of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

a) Find the matrix P of the orthogonal projection onto V.

b) Find the matrix P' of the orthogonal projection on to $V^{\perp}$.