Name:

MWF 9 Fabian Haiden
MWF 10 Ziliang Che
MWF 10 Jeremy Hahn
MWF 11 Rosalie Belanger-Rioux
MWF 11 Yu-Wen Hsu
MWF 12 Peter Garfield
TThu 10 Oliver Knill
TThu 11:30 Alex Perry
TThu 11:30 Rong Zhou

- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

1)	T	F	The geometric multiplicity of the eigenvalue 1 of the shear $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is equal to 2.
2)	T	F	Two symmetric 4×4 matrices are similar if their trace, determinant and eigenvalues are the same.
3)	T	F	Every orthogonal 3×3 matrix satisfies $A = A^{-1}$.
4)	T	F	If A is a 5×5 matrix and $A^2 = I_5$ and $A = A^T$, then A is similar to an orthogonal matrix.
5)	T	F	The trace of a 4×4 matrix A does not change under row reduction.
6)	T	F	Every 3×3 matrix A for which $T(x) = Ax$ is a reflection about a plane can be diagonalized.
7)	T	F	The recursion $x_{t+1} = x_t + 3x_{t-1} + 1$ can be written as a vector equation $\vec{v}_{t+1} = A\vec{v}_t$ for a 2×2 matrix A and vector $\vec{v}_t = [x_t, x_{t-1}]^T$.
8)	T	F	The determinant of a reflection matrix is either 1 or -1.
9)	T	F	The nullity of a $n \times n$ matrix A is the same as the rank of A^T.
10)	T	F	For any 3×3 matrix, we have $\det(A^5) = \det(A)^5$.
11)	T	F	A discrete dynamical system $\vec{v}_{t+1} = A\vec{v}_t$ defined by a 2×2 matrix A is asymptotically stable if $A^8 = 0$.
12)	T	F	A 2×2 matrix for a reflection about a line always has trace 0.
13)	T	F	The matrix $A = \begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 3 & 3 \\ 0 & 4 \end{bmatrix}$.
14)	T	F	If A is a 5×3 matrix of rank 3, then the least square solution of $Ax = b$ is unique.
15)	T	F	If A is asymptotically stable, then A^T is asymptotically stable.
16)	T	F	If A is similar to C and B is similar to D, then $A + B$ is similar to $C + D$.
17)	T	F	If A is similar to A^{-1} then A is the identity.
18)	T	F	The sum of the geometric multiplicities of a $n \times n$ matrix is always n.
19)	T	F	If A has the same determinant as B^2, then B has the same determinant as A^2.
20)	T	F	If A is a non-invertible 3×3 matrix with trace 6 and such that 5 is an eigenvalue, then 1 is another eigenvalue of A.

Total

Problem 2) (10 points) No justifications are needed.

a) (2 points) Which matrices have the property that the system $x(t+1) = Ax(t)$ is asymptotically stable (for which we just write "stable")?

Matrix	stable	not stable
$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$		
$\begin{bmatrix} 0.9 & 1 \\ 0 & 0.9 \end{bmatrix}$		
$\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$		
$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$		

b) (2 points) Which identities hold for a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with eigenvalues α, β.

Identity	always true	not always true
$\text{tr}(A) = \alpha + \beta$		
$\det(A) = \alpha\beta$		
$\det(A) = \text{tr}(A)$		
$\det(A - \lambda I_2) = (\alpha - \lambda)(\beta - \lambda)$		

c) (2 points) Which of the following complex numbers are real? Remember that we defined $w^z = e^{z \log(w)}$ and $\log(z) = \ln|z| + i \arg(z)$ with $0 \le \arg(z) < 2\pi$ for any complex numbers $w \neq 0, z \neq 0$ and where $\arg(z)$ is the angle so that $z = |z|e^{i \arg(z)}$.

Number	is real	is not real
1^i		
2^i		
i^2		
i^i		

d) (2 points) Which types of matrices are always diagonalizable over the real numbers?

Type of matrix	diagonalizable	not diagonalizable
orthogonal		
projection		
symmetric matrix		
all geometric multiplicities are 1		

e) (2 points) If S is a 3×3 matrix whose columns are given by an eigenbasis of a matrix A which has eigenvalues $1, 2, 3$, then

Statement	is always true	is not always true
S is invertible		
S is a projection		
S is orthogonal		
S is symmetric		

Problem 3) (10 points) No justifications are needed

a) (2 points) One of the following formulas gives the projection onto the image of A. Which one?

$A(A^TA)^{-1}A^T$	
$A^T(A^TA)^{-1}A$	
$A(AA^T)^{-1}A^T$	
$A^T(AA^T)^{-1}A^T$	
$(AA^T)^{-1}A^T$	
$(A^TA)^{-1}A^T$	

b) (2 points) Which of the following matrices have real eigenvalues?

$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$	

c) (2 points) For which of the following matrices does the identity matrix appear either for Q or for R in the QR factorization?

$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$	
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$	
$A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$	

d) (2 points) Which of the following matrices is useful to find the closed form solution for the recursion $x_{t+1} = 5x_t - 4x_{t-1}$?

$A = \begin{bmatrix} 5 & -4 \\ 1 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 5 & -4 \\ 0 & 1 \end{bmatrix}$	
$A = \begin{bmatrix} 5 & 0 \\ -1 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 5 & 0 \\ -1 & 4 \end{bmatrix}$	

e) (2 points) Two of the following statements are part of the spectral theorem. Which ones?

A symmetric matrix has an orthonormal eigenbasis	
An orthogonal matrix is diagonalizable over the reals	
A real matrix has real eigenvalues	
A symmetric matrix has real eigenvalues	
A matrix with distinct eigenvalues has an orthonormal eigenbasis	

Problem 4) (10 points)

Find the function $y = ax + bx^3$ which provides the best fit for the data

x	y
1	2
-2	1
-1	2
0	2

.

Problem 5) (10 points)

The recursion $x_{t+1} = 3x_t - 2x_{t-1}$ with $x_0 = 0, x_1 = 1$ is an example of a **Lucas sequence**. It starts with $0, 1, 3, 7, 15, 31, 63, ...$ Recall that we can write the problem as a discrete dynamical system

$$\vec{v}_{t+1} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \vec{v}_t \; .$$

Find a closed form solution for $\vec{v}_t$ if the initial condition is

$$\vec{v}_0 = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; .$$

Isabel Vogt and **Jesse Silliman** recently showed that the given Lucas sequence contains no perfect powers like for example 7^3. Isabel has been a Harvard math concentrator here. The preprint is here http://arxiv.org/pdf/1307.5078v2.pdf

Problem 6) (10 points)

a) (5 points) We look at the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix} \; .$$

Find all the eigenvectors and the characteristic polynomial f_A of A.

b) (3 points) You know that A is similar to a diagonal matrix B. Write down this matrix B.

c) (2 points) Also write down the matrix S which satisfies $B = S^{-1}AS$.

Problem 7) (10 points)

a) (5 points) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 3 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} .$$

Note that it is of the form $Q + 3Q^{-1} + 2I_7$, where Q is a matrix of a type you have seen a lot.

b) (5 points) Determine all the eigenvectors of that matrix using $A = Q + 3Q^{-1} + 2I_7$.

Remark: You do not have to simplify the expressions you get for the eigenvalues and eigenvectors.

Problem 8) (10 points)

a) (2 points) What is the determinant of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 4 & 5 \\ 1 & 2 & 4 & 4 & 5 \\ 1 & 2 & 3 & 5 & 5 \\ 1 & 2 & 3 & 4 & 6 \end{bmatrix}$?

b) (2 points) Find the determinant of the matrix $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$.

c) (3 points) To commemorate the "Go" match between computer and human from March 2016, we pose a determinant problem for a 19×19 matrix: evaluate

$$\begin{bmatrix}
\boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \boxed{1} & 0 & \boxed{1} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0
\end{bmatrix}$$

d) (3 points) find the determinant of $\begin{bmatrix} 7 & 2 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 2 & 7 \end{bmatrix}$.

Problem 9) (10 points)

a) (4 points) Use Gram-Schmidt to find an orthonormal basis of the image of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} .$$

b) (3 points) Let Q be the matrix which contains this orthonormal basis. What is the 2×2 matrix $Q^T Q$?

c) (3 points) What is the projection $QQ^T\vec{v}$ on the image of A if $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.