

Name:	
MWF9 Fabian Haiden	<ul style="list-style-type: none"> <li>• Start by writing your name in the above box and check your section in the box to the left.</li> <li>• Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.</li> <li>• Do not detach pages from this exam packet or un-staple the packet.</li> <li>• Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.</li> <li>• No notes, books, calculators, computers, or other electronic aids can be allowed.</li> <li>• You have 90 minutes time to complete your work.</li> </ul>
MWF10 Ziliang Che	
MWF10 Jeremy Hahn	
MWF11 Rosalie Belanger-Rioux	
MWF11 Yu-Wen Hsu	
MWF12 Peter Garfield	
TThu10 Oliver Knill	
TThu11:30 Alex Perry	
TThu11:30 Rong Zhou	

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

- 1)  T  F A linear system with 2 equations and 3 unknowns has either infinitely many or no solutions.

**Solution:**

Geometrically, this is an intersection of two planes. They are either the same plane or intersect in a line.

- 2)  T  F If  $S$  is an invertible matrix which contains the vectors  $\vec{v}_1, \dots, \vec{v}_n$  as columns, then  $\vec{v}_1, \dots, \vec{v}_n$  is a basis of  $\mathbf{R}^n$ .

**Solution:**

This is an important property of a basis in  $\mathbf{R}^n$ . Row reduction shows that the image of  $A$  is the entire space  $\mathbf{R}^n$  so that the vectors span. By the dimension formula the kernel is zero dimensional. Having a trivial kernel is equivalent to the fact that the vectors are linearly independent.

- 3)  T  F If  $A, B$  are given  $n \times n$  matrices, then the formula  $(A - B)(A + B) = A^2 - B^2$  holds.

**Solution:**

Matrix multiplication is not commutative.

- 4)  T  F Suppose  $A$  is an  $m \times n$  matrix, where  $n < m$ . If the rank of  $A$  is  $m$ , then there is a vector  $y \in \mathbf{R}^m$  for which the system  $Ax = y$  has no solutions.

**Solution:**

The matrix has  $m$  rows and  $n$  columns. The rank can be maximally  $n$ . The statement that the rank of  $A$  is  $m$  is not possible. Since this was a false assumption, the conclusion is true. You might have come to the right conclusion using other reasoning: there is at least one row for which all entries are 0 entries. After row reduction of the augmented matrix  $[A|b]$ , we can end up with a last row  $[0, \dots, 0|-1]$  which means, we have no solution.

- 5)  T  F The matrix  $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$  is invertible.

**Solution:**

You see immediately that two rows are parallel. The image is a plane.

- 6)  T  F The rank of a lower-triangular matrix equals the number of non-zero entries along the diagonal.

**Solution:**

The matrix  $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$  has all zeros in the diagonal and is rank 1.

- 7)  T  F The row reduced echelon form of a  $3 \times 3$  matrix of rank 2 is one of the following  $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$  or  $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .

**Solution:**

We can also have the example  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

- 8)  T  F The matrix  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  is a shear.

**Solution:**

A shear has the property that  $Av = v$  for some vector  $v$ . This would mean that  $A - I_2$  has a kernel. But  $A - I = \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}$  is invertible.

- 9)  T  F For any matrix  $A$ , one has  $\dim(\ker(A)) = \dim(\ker(\text{rref}(A)))$ .

**Solution:**

Row reduction does not change the kernel.

- 10)  T  F If  $A$  is a square matrix for which  $\ker(A)$  is included in  $\text{im}(A)$ , then  $A$  is not invertible.

**Solution:**

Also for invertible  $A$ , the kernel is included in the image.

- 11)  T  F There exists an invertible  $3 \times 3$  matrix, for which 7 of the 9 entries are  $\pi$ .

**Solution:**

Example: 
$$\begin{bmatrix} \pi & \pi & \pi \\ \pi & 0 & \pi \\ \pi & \pi & 0 \end{bmatrix}.$$

- 12)  T  F If  $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ , then  $A^T A$  is the projection onto a line.

**Solution:**

Yes, it is a special case of  $Q Q^T = P$  being a projection.

- 13)  T  F If  $A$  and  $B$  are  $n \times n$  matrices, then  $AB$  is invertible if and only if both  $A$  and  $B$  are invertible.

**Solution:**

Indeed, the inverse is  $(AB)^{-1} = B^{-1}A^{-1}$ . To see the other direction: if  $AB$  is invertible, then  $\text{rank}(AB) = n$ . Since  $\text{im}(AB)$  is contained in  $\text{im}(A)$ ,  $\dim(\text{im}(AB)) \leq \dim(\text{im}(A))$ . Because the dimension of the image is just the rank,  $\text{rank}(AB) \leq \text{rank}(A)$ , so  $\text{rank}(A) = n$ , which shows that  $A$  is invertible. Then,  $B$  is invertible since  $B$  can be written as  $A^{-1}(AB)$ , which is a product of two invertible matrices.

- 14)  T  F There exist matrices  $A, B$  such that  $A$  has rank 4 and  $B$  has rank 7 and  $AB$  has rank 5.

**Solution:**

The rank of  $AB$  is smaller or equal then the rank of  $A$  because the image of  $AB$  is contained in the image of  $A$ .

- 15)  T  F There exist matrices  $A, B$  such that  $A$  has rank 2 and  $B$  has rank 7 and  $AB$  has rank 1.

**Solution:**

Indeed this is possible even for diagonal matrices with entries 0 and 1.

- 16)  T  F If for an invertible matrix  $A$  one has  $A^2 = A$ , then  $A = I_n$ .

**Solution:**

Multiply the equation  $A^2 = A$  with  $A^{-1}$ .

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 If an invertible matrix  $A$  satisfies  $A^2 = I_2$ , then  $A = I_2$  or  $A = -I_2$ .

**Solution:**

You can have  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  for example.

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 The matrix  $\begin{bmatrix} c-1 & -1 \\ 2 & c+1 \end{bmatrix}$  is invertible for every real number  $c$ .

**Solution:**

The matrix is invertible as long as the rank is 2. Since we know the image is spanned by the columns, we just need to determine if the columns are scalar multiples of each other. If so, then we must have

$$\begin{bmatrix} c-1 \\ 2 \end{bmatrix} = (1-c) \begin{bmatrix} -1 \\ c+1 \end{bmatrix}.$$

This means that  $2 = 1 - c^2$ , or  $c^2 = -1$ , which is impossible. Therefore, the columns are linearly independent, and the matrix is invertible.

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 For  $2 \times 2$  matrices  $A$  and  $B$ , if  $AB = 0$ , then either  $A = 0$  or  $B = 0$ .

**Solution:**

Take  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ .

- 20) 

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 If  $T$  is a rotation in space with an angle  $\pi/6$  around the  $z$  axes, then the linear transformation  $S(x) = T(x) - x$  is invertible.

**Solution:**

The vector  $e_3$  has the property that  $S(e_3) = 0$ .

Problem 2) (10 points)

Match each of matrices with one of the geometric descriptions below. You don't have to give explanations.

Matrix	Enter A-H here.
a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
b) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	
d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	

Matrix	Enter A-H here.
e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	
f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
g) $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
h) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	

- A) Shear along a plane.
- B) Projection onto a plane.
- C) Rotation around an axes.
- D) Reflection at a point.
- E) Projection onto a line.
- F) Reflection at a plane.
- G) Reflection at a line.
- H) Identity transformation.

**Solution:**

- a) = B)
- b) = C)
- c) = D)
- d) = F)
- e) = A)
- f) = H)
- g) = E)
- h) = G)

**Problem 3) (10 points)**

- a) Write the matrix  $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$  as a product of a rotation and a dilation.
- b) What is the length of the vector  $\vec{v} = A^{100}e_1$ , where  $e_1$  is the first basis vector?
- c) In which direction does the vector  $\vec{v}$  point?
- d) Find a matrix  $B$  such that  $B^2 = A$ .

**Solution:**

- a) The matrix is a rotation dilation matrix, a rotation by  $\pi/4$  and scaling by  $\sqrt{2}$ .
- b)  $A^{100}$  is a composition of a scaling by a factor  $\sqrt{2}^{100} = 2^{50}$  and rotation by  $\pi$ . So,  $A^{100} = \begin{bmatrix} -2^{50} & 0 \\ 0 & -2^{50} \end{bmatrix}$ . The answer is  $2^{50}$ .
- c) It points to  $-e_1$ : after each 8 rotations, we are back to the initial position, so also after 96 rotations. The additional 4 rotations turn the vector to  $-e_1$ .
- d) A rotation by angle  $\pi/8$  and scaling  $2^{1/4}$  gives a rotation-dilation matrix with  $a = 2^{1/4} \cos(\pi/8)$ ,  $b = 2^{1/4} \sin(\pi/8)$ . The matrix is

$$A = 2^{1/4} \begin{bmatrix} \cos(\pi/8) & -\sin(\pi/8) \\ \sin(\pi/8) & \cos(\pi/8) \end{bmatrix}.$$

**Problem 4) (10 points)**

Let  $A$  be a  $3 \times 3$  matrix such that  $A^2 = 0$ . That is, the product of  $A$  with itself is the zero matrix.

- a) Verify that  $\text{Im}(A)$  is a subspace of  $\text{ker}(A)$ .

- b) Can  $\text{ran}(A) = 2$ ? If yes, give an example.
- c) Can  $\text{ran}(A) = 1$ ? If yes, give an example.
- d) Can  $\text{ran}(A) = 0$ ? If yes, give an example.

**Solution:**

a) If  $y$  is in the image, then  $y = A(x)$  and  $A(y) = A^2x = 0$ .

b) No: If  $\dim(\text{im}(A)) = 2$ , then  $\dim(\ker(A)) = 1$  and  $\dim(\ker(A^2))$  has maximal 2 dimensions so that the rank of  $A^2$  would be at least 1. A direct proof: If the rank of  $A$  is 2, then  $\dim(\text{im}A) = 2$ . By the rank - nullity theorem,  $\dim(\ker(A)) = 1$ . On the other hand, by part (a),  $\text{im}(A)$  is a subspace of  $\ker(A)$ , so  $\dim(\text{im}(A))$  cannot be bigger than  $\dim(\ker(A))$ .

c) Yes:  $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

d) Yes:  $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

Problem 5) (10 points)
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Let  $b, c$  be arbitrary numbers. Consider the matrix  $A = \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -c \\ -b & c & 0 \end{bmatrix}$ .

- a) Find  $\text{rref}(A)$  and find a basis for the kernel and the image of  $A$ .
- b) For which  $b, c$  is the kernel one dimensional?
- c) Can the kernel be two dimensional?

**Solution:**

a)  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & -b \\ 0 & 0 & 0 \end{bmatrix}$ . We see that the first two columns are pivot columns. The

first two columns of  $A$  form a basis of the image. To find a basis for the kernel, introduce a free variable  $z = t$  for the last (redundant) column. We get  $x - ct = 0, y - bt = 0, z = t$

which gives  $(x, y, z) = (ct, bt, t) = (c, b, 1)t$ . So  $\begin{bmatrix} c \\ b \\ 1 \end{bmatrix}$  is a basis for the kernel.

b) The kernel is always one-dimensional.

c) No, since we know from part (b) that the kernel is always one-dimensional.

Problem 6) (10 points)

Consider the matrix  $A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix}$ .

a) Use a series of elementary Gauss-Jordan row operations to find the reduced row echelon form  $\text{rref}(A)$  of  $A$ . Do only one elementary operations at each step.

b) Find the rank of  $A$ .

c) Find a basis for the image of  $A$ .

d) Find a basis for the kernel of  $A$ .

**Solution:**

a) We end up with  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) The rank is 2, then number of pivot columns.

c) A basis of the image are the first two column vectors in  $A$ .

d) A basis of the kernel is  $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ .

Problem 7) (10 points)

Let  $A$  be a  $2 \times 2$  matrix and  $S = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ . We know that  $B = S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . Find  $A^{2003}$ .

**Solution:**

1. Solution. See inductively that  $B^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ .

2. Solution Write  $B = (I_2 + C)$ , note that  $C^2 = 0$  and remember  $(1+x)^n = 1 + nx + \dots + x^n$ .  $B^{2003} = (1 + 2003C)$  so that  $A^{2003} = S(1 + 2003C)S^{-1} = 1 + 2003(SCS^{-1})$ . Because

$S^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$  and  $2003SCS^{-1} = 2003 \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}$  we have - would have been easier

to add up 2000 year ago... -  $A^{2003} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2003 \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix} = \begin{bmatrix} -12017 & 8012 \\ -18027 & 12019 \end{bmatrix}$ .

Problem 8) (10 points)

Let  $A$  be a  $5 \times 5$  matrix. Suppose a finite number of elementary row operations reduces  $A$  to

the following matrix  $B = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ .

a) Find a basis of the kernel of  $A$ .

b) Suppose the elementary row operations used in reducing  $A$  to  $B$  are the following:

- i) Add row 2 to row 3.
- ii) Swap row 2 and row 4.
- iii) Multiple row 4 by  $1/2$ .
- iv) Subtract row 1 from row 5.

Find a basis of the image of  $A$ .

**Solution:**

a) The matrix  $B$  and the matrix  $A$  have the same reduced row echelon form.

$$\text{rref}(A) = \text{rref}(B) = B = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ If we call the variables } x_1, \dots, x_5, \text{ this}$$

matrix corresponds to the equations  $x_1 - x_4 = 0$ ,  $x_2 = 0$ ,  $x_3 = 0$ , and  $x_5 = 0$ .  $x_4$  is a free variable and can have any value  $t$ , so the elements of the kernel are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus,  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$  is a basis of  $\ker A$ .

b) We reverse the steps:

iv inverse) Add row 1 to row 5

iii inverse) multiply row 4 by 2

ii inverse) swap row 2 and row 4

i inverse) subtract row 2 from row 3

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

Pick columns 1,2,3,5 of the last matrix.

**Problem 9) (10 points)**

a) Find a basis for the plane  $x + 2y + z = 0$  in  $\mathbf{R}^3$ .

b) Find a  $3 \times 3$  matrix which represents (with respect to the standard basis) a linear transformation with image the plane  $x + 2y + z = 0$  and with the kernel the line  $x = y = z$ .

**Solution:**

a) The solution is  $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$  since

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

b) The image of any matrix is spanned by the columns of the matrix. Since the plane  $x + 2y + z = 0$  is spanned by the vectors  $\vec{v}_1$  and  $\vec{v}_2$  from part (a), let's use  $\vec{v}_1$  and  $\vec{v}_2$  as the first two columns of our matrix. Then, our matrix looks like  $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$  for some vector

$\vec{v}_3$ . Since we want the kernel of this matrix to be spanned by  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , we need

$$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is, we want  $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$ , so  $\vec{v}_3 = -\vec{v}_1 - \vec{v}_2$ . Thus, the matrix is

$$\begin{bmatrix} -2 & -1 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Problem 10) (10 points)
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The set of  $2 \times 2$  matrices form a linear space like vectors. One can add them, scale them and the zero matrix is the zero element. If you like you could stack the two column vectors onto each other to get a vector of length 4. But this is not really needed. Let  $T$  be the linear map from linear space  $X$  of  $2 \times 2$  matrices to the real line which assigns to the matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  its trace  $T(A) = a + d$ .

- a) (3 points) What is the dimension of the image of  $T$ ?
- b) (3 points) What is the dimension of the kernel of  $T$ ?
- c) (3 points) Find an explicit nonzero matrix in the kernel of  $T$ .
- d) (2 points) Is the transformation  $T(A) = \det(A) = ad - bc$  a linear map from  $X$  to  $\mathbb{R}$ ?

**Solution:**

- a) The map is  $T(a, b, c, d) = a + d$ . The image is the entire line. The dimension of the image is 1.
- b) By the dimension formula, the dimension of the kernel is 3.
- c) The reflection about the  $y$ -axes.
- d) No, it is nonlinear. The determinant of  $2A$  is 4 times  $\det(A)$ .