Name:

MWF9 Fabian Haiden	• Start by writing your name in the above box and check your section in the box to the left.
MWF10 Ziliang Che	• Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
MWF10 Jeremy Hahn	
MWF11 Rosalie Belanger-Rioux	
MWF11 Yu-Wen Hsu	• Do not detach pages from this exam packet or un-staple the packet.
MWF12 Peter Garfield	• Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
TThu10 Oliver Knill	
TThu11:30 Alex Perry	• No notes, books, calculators, computers, or other electronic aids can be allowed.
TThu11:30 Rong Zhou	• You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) [T][F]   The rank of A^{-1} is always equal to the rank of A if A is an invertible matrix.

2) [T][F] $\text{rank}(A - B) = \text{rank}(A) - \text{rank}(B)$ for all 2×2 matrices.

3) [T][F]   The row reduced echelon form of an invertible 3×3 matrices is invertible.

4) [T][F]   Given 3 vectors in R^5, then their span forms a linear space.

5) [T][F]   A system of linear equations has either 0, 1 or ∞ many solutions.

6) [T][F]   A reflection in the plane at the x axes is similar to the reflection at the y axes.

7) [T][F]   Every basis of R^3 contains exactly 3 vectors in it.

8) [T][F]   A 4×4 matrix can have $\dim(\text{im}(A)) = \dim(\ker(A))$.

9) [T][F]   The rank of a 7×3 matrix can be 4.

10) [T][F]   If $\{v_1, v_2, v_3, v_4\}$ is a set of vectors spanning a linear subspace V of $\mathbf{R}^9$, then $\dim(V) \geq 4$.

11) [T][F]   If A is a 7×5 matrix, then the dimension of $\ker(A)$ is at least 2.

12) [T][F]   The difference $A - B$ of 2 invertible 5×5 matrices A, B is invertible.

13) [T][F]   If $A\vec{x} = \vec{0}$ has a nonzero solution, where A is a 4×4 matrix, then $\text{rank}(A) \leq 3$.

14) [T][F]   If $\vec{b}$ is in $\text{im}(A)$, then $A\vec{x} = \vec{b}$ has exactly one solution.

15) [T][F]   If A and B are 2×2 matrices and $A \cdot B$ is the identity matrix 1_2, then A and B are both invertible.

16) [T][F]   If $\vec{v}$ is a nonzero vector in the kernel of A, then $\vec{v}$ is perpendicular to every row vector of A.

17) [T][F]   If $AB = I_2$ for an 2×3 matrix A and B is a 3×2 matrix, then $BA = I_3$.

18) [T][F]   If a 2×2 matrix different from the identity is its own inverse then it is a reflection at a line.

19) [T][F]   The set of vectors (x, y) in R^2 such that $|x| + y = 0$ is a linear subspace of R^2.

20) [T][F]   Given a rotation dilation matrix A and a projection matrix B. Then the intersection of the image of A and the kernel of B is a linear space.

[ ]   Total

a) (5 points) Which of the following matrices are in row reduced echelon form?

Matrix	is in row reduced echelon form
$\begin{array}{cccccc} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$	
$\begin{array}{cccccc} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}$	
$\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}$	
$\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}$	
$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}$	

b) (5 points) Check the matrices which are invertible:

Matrix	invertible
$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}$	
$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}$	
$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 4 & 1 & 0 \end{array}$	
$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}$	
$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}$	

a) (3 points) Which of the following matrices either perform a rotation dilation or a reflection dilation? Check the corresponding boxes (it is also possible that both cases are unchecked):

Matrix	rotation dilat.	reflection dilat.	Matrix	rotation dilat.	reflection dilat.
$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$			$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$		
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$			$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$		

b) (2 points) Which of the following sets are linear spaces?

The space of all ...	Check	The space of all ...	Check
all (x, y) satisfying $x^2 + y^2 = 1$		$(x, y, z) \in \mathbf{R}^3$ satisfying $2x + y - 4z = 1$	
all (x, y) satisfying $x^2 = y$		the set of rational numbers in R	

c) (5 points) Match the matrices with the action of the transformation which maps a shape in the domain $\mathbb{R}^2$ into a shape of the codomain $\mathbb{R}^2$.

A-F	domain	codomain	A-F	domain	codomain

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

Problem 4) (10 points)

a) (5 points) Find a basis of the image of the following **chess matrix**:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

b) (5 points) Find a basis for the linear subspace of all vectors in $\mathbf{R}^4$ which are perpendicular to the columns of the matrix

$$A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}.$$

Problem 5) (10 points)

a) (5 points) Invert the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

by using row reduction on an augmented 4×8 matrix

b) (5 points) Find a basis for the linear space of vectors perpendicular to the kernel of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Problem 6) (10 points)

a) (2 points) Find the angle between the vectors $x = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$.

b) (2 points) Find a matrix A whose image is spanned by the two vectors x, y.

c) (4 points) Find a matrix B whose image is the space of all vectors perpendicular to the two vectors.

d) (2 points) What is the relation between the image of B and the kernel of the transpose of A?

Problem 7) (10 points)

a) (6 points) Find a basis of the space V of all vectors perpendicular to the three vectors

$$\{v_1, v_2, v_3\} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \} \,.$$

b) (4 points) Use a) to find a basis for R^4 which contains v_1, v_2, v_3.

Problem 8) (10 points)

a) (5 points) The projection-dilation matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ in the basis

$$\mathcal{B} = \{v_1, v_2, v_3\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$$

is given by a matrix B. Find this 3×3 matrix B.

b) (5 points) A linear transformation T satisfies

$$T(v_1) = v_2, T(v_2) = v_3, T(v_3) = v_1$$

where v_1, v_2, v_3 are given in a). Find the matrix R implementing this transformation in the standard basis.

Problem 9) (10 points)

a) (5 points) Find A^{10} where $A = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

b) (5 points) Find a 2×2 matrix X satisfying $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} X \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$

Problem 10) (10 points)

The data points $(1, 2), (2, 2), (-3, -4)$ in the plane relate two vectors $X = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and

$Y = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$.

a) (2 points) Find the length of X and Y. (Since the vectors are already centered these are up to a constant the standard deviation)

b) (2 points) Find the correlation coefficient, the cos of the angle between X and Y.

c) (4 points) The regression line $y = ax$ is given by the formula $a = X \cdot Y / |X|^2$. Find a.

d) (2 points) What can you deduce from c): is the angle between X and Y acute or obtuse?