

## Homework 21: Complex Eigenvalues

This homework is due on Wednesday, March 30, respectively on Thursday, March 31, 2016.

- 1 a) For  $z = 2 + 4i$ . Find  $7z + z^3$ .
- b) The log of a nonzero complex number  $re^{i\theta} = r \cos(\theta) + ir \sin(\theta)$ , where  $r > 0$  and  $0 \leq \theta < 2\pi$ , is given by  $\log(re^{i\theta}) = \log r + i\theta$ . Find the logarithm of  $e \cdot i$ . ("Ei" is the product of  $e$  and  $i$ . It means "Egg" in German. It was just easter!)
- c) Using logarithms we can define  $w^z = e^{z \log w}$ . What is  $i^i$ , the "eye for an eye" number?

- 2 Express  $\cos(4\theta)$  and  $\sin(4\theta)$  as polynomials in  $\cos(\theta)$ ,  $\sin(\theta)$ .

- 3 a) Find all the complex eigenvalues of the matrix  $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ .

(Note that real numbers are a subset of complex numbers and also complex. Similarly as rational numbers are real numbers too).

- b) Verify that if  $\lambda$  is an eigenvalue,  $\vec{v} = \begin{bmatrix} \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$  is an eigenvector.

- 4 Find the eigenvalues and eigenvectors of the matrix  $\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 1 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$ .

Hint: Do you recognize  $A - 5I_6$ ?

- 5 The matrix  $A = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.8 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.0 \end{bmatrix}$  is called a Markov matrix: in every column the entries add up to 1, so each column can be in-

terpreted as a probability distribution.

a) Verify that  $A^T$  has the eigenvector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  with eigenvalue 1.

b) Why does  $A$  also have an eigenvalue 1? Find all other eigenvalues.

c) Find all eigenvectors of  $A$ .

## Complex eigenvalues

Numbers of the form  $z = a+ib$  are called complex numbers. One can add and multiply these numbers as real numbers, just keeping in mind that  $i^2 = -1$ . Euler found  $e^{i\theta} = \cos(\theta) + i \sin(\theta)$  which in the special case  $\theta = \pi$  gives  $e^{i\pi} + 1 = 0$ , which is by far voted the most beautiful formula in whole of mathematics as it combines  $e, \pi, i, 1$  and  $0$ . As  $e$  is part of analysis and  $\pi$  is part of geometry and  $0$  is the additive neutral element and  $1$  the multiplicative neutral element, this identity combines analysis, geometry and algebra. The Euler identity leads to **de Moivre formulas** like  $(\cos(\theta) + i \sin(\theta))^3 = (e^{i\theta})^3 = e^{i3\theta} = \cos(3\theta) + i \sin(3\theta)$ . So that  $\cos^3(\theta) - \cos(\theta) \sin^2(\theta) = \cos(3\theta)$  and  $\cos^2(\theta) \sin(\theta) - \sin^3(\theta) = \sin(3\theta)$ . Eigenvalues of matrices can become complex as the rotation-dilation matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

shows which has the eigenvalues  $a \pm ib$ . The fundamental theorem of algebra assures that the sum of the algebraic multiplicities of all eigenvalues of a  $n \times n$  matrix is equal to  $n$ .