

## Homework 19: Eigenspaces

This homework is due on Friday, March 25, respectively on Tuesday, March 29, 2016.

- 1 Find all eigenvalues and eigenvectors for the following matrix  $A =$

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

What are the algebraic and geometric multiplicities? As a hint, we tell you that the eigenvalues 0, 1, 1, 3 appear. Use the trace trick to the 5th eigenvalue.

- 2 Assume that a  $2 \times 2$  matrix has trace  $-5$  and determinant 6. Find its eigenvalues and find a non-diagonal matrix which realizes the situation.
- 3 a) Verify that  $A$  and  $A^T$  have the same eigenvalues.  
 b) Assume  $A$  is invertible. What is the relation between the eigenvalues of  $A$  and  $A^{-1}$ ?



- 4 The vector  $A^n b$  gives pollution levels in the Silvaplana, Sils and St Moritz lake  $n$  weeks after an oil spill. The matrix is  $A =$
- $$\begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$
- and  $b = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$  is the initial pollution level. Find a closed form solution for the pollution after  $n$  weeks.

5 1) Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 6 & 6 & 0 \\ 1 & 14 & 36 & 24 \end{bmatrix}.$$

b) Find the eigenvectors of  $A^3$ , where  $A$  is the previous matrix.

c) Find the eigenvectors of  $(A^T)^{-1}$ , where  $A$  is the previous matrix.

## Eigenspaces

A nonzero vector  $\vec{v}$  is called an **eigenvector**, if  $Av = \lambda v$  for some  $\lambda$ . The set of eigenvectors is called the **eigenspace**  $E_\lambda$ . It is the kernel of  $A - \lambda I_n$ . The dimension of the eigenspace is called the **geometric multiplicity** of  $\lambda$ . There is a general result which tells that the geometric multiplicity of  $\lambda$  is always smaller or equal to the algebraic multiplicity.

Recall that  $A$  is similar to  $B$  if there exists an invertible  $S$  such that  $B = S^{-1}AS$ . If  $A$  and  $B$  are similar, then they have the same characteristic polynomial, the same eigenvalues and algebraic multiplicities as well as the same geometric multiplicities. Similar matrices also have the same trace  $\text{tr}(A) = \lambda_1 + \dots + \lambda_n$  as well as determinant  $\det(A) = \lambda_1 \cdots \lambda_n$ . These formulas hold in general if we allow the eigenvalues  $\lambda_i$  to be complex. More on complex eigenvalues next week.