

Homework 14: Orthogonal transformations

This homework is due on Monday, March 7, respectively on Tuesday, March 8, 2016.

- 1 Determine from each of the following matrices whether they are orthogonal:

$$\begin{aligned} \text{a) } & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} / 2, \text{ b) } \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} / 2, \text{ c) } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ \text{d) } & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ e) } \begin{bmatrix} \cos(1) & \sin(1) & 0 & 0 \\ -\sin(1) & \cos(1) & 0 & 0 \\ 0 & 0 & \cos(2) & \sin(2) \\ 0 & 0 & \sin(2) & -\cos(2) \end{bmatrix}. \end{aligned}$$

- 2 If A, B are orthogonal, then

a) Is $A + B$ orthogonal? b) Is $3A$ orthogonal? c) Is A^T orthogonal? d) B^{-1} is orthogonal? e) Is $B^{-1}AB$ orthogonal?

- 3 a) Matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ can be multiplied and the result is of the same form. These rotation dilation matrices are also called “complex numbers”! Which of these matrices plays the role of $i = \sqrt{-1}$, that is, which of them has the property that $A^2 = -1$ (where -1 means $-I_2$)?

b) Figure out the formula for the multiplication $(a + ib)(c + id)$ of complex numbers by looking at the product $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$.

c) If you draw complex numbers $a + ib, c + id$ as vectors, what is the multiplication geometrically?

- 4 Mathematicians for a long time looked for higher dimensional analogues of the complex numbers. Matrices of the form $A(p, q, r, s) =$

$$\begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$$
 are called **quaternions**. They were invented by Hamilton.

- a) Find a basis for the set of all the matrices above.
- b) Check that every matrix in the unit sphere $p^2 + q^2 + r^2 + s^2 = 1$ in the four dimensional space of quaternions corresponds to an orthogonal matrix.

- 5 a) Explain why the identity matrix is the only $n \times n$ matrix that is orthogonal, upper triangular and has positive entries on the diagonal. b) Show that the QR factorization of an invertible $n \times n$ matrix A is unique. That is, if $A = Q_1R_1$ and $A = Q_2R_2$ are two factorizations, argue why $Q_1 = Q_2$ and $R_1 = R_2$.

Orthogonal transformations

The transpose $A_{ij}^T = A_{ji}$ satisfies $(AB)^T = B^T A^T$ and $(A^T)^T = A$. The rank of the transpose is the same as the rank of A . An $n \times n$ matrix A is **orthogonal** if $A^T A = 1 = 1_n$. The linear transformation of an orthogonal matrix is called an **orthogonal transformation**. It preserves length and angle. The column vectors of an orthogonal matrix forms an orthonormal basis. The product of two orthogonal matrices is orthogonal. The inverse A^{-1} is orthogonal and given by A^T . Examples of orthogonal transformations are rotations or reflections or the identity matrix.