Homework 9: Dimension

This homework is due on Monday, February 22, respectively on Tuesday, February 23, 2016.

1 Determine the rank and nullity of the following matrix and verify that the rank nullity theorem holds in this case:

$$A = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

2 a) In each of the 5 cases $k = 1, 2, 3, 4, 5$; give a 4×5 matrix A for which the dimension of the kernel of A is k.

b) Is there a 4×5 matrix whose kernel has dimension 0? Explain why or why not.

3 Find out whether the following set of vectors related to prime numbers forms a basis of R^4.

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 11 \\ 13 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \\ 13 \\ 17 \end{bmatrix} \right\}$$

4 The following matrices are meant to look like letters:

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, O = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

a) Find four letters which have the same kernel.

b) Find three letters which have the same image.

5 Consider the subspace V consisting of all vectors in R^4 which are

perpendicular to both $v = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

Find a basis for V.

Basis

If V is a linear space and $\{v_1, .., v_n\}$ is a basis, then n is the **dimension** of V. If $\vec{v}_1, \ldots, \vec{v}_p$ are linearly independent in V and $\vec{w}_1, \ldots, \vec{w}_q$ span V then $p \leq q$. The dimension of the image of A is called the **rank** of A. The dimension of the kernel of A is called the **nullity** of A. The rank-nullity theorem tells that the sum of the rank and the nullity is equal to the number of columns of A. It's easy to see why this is true if you remember that the rank of A is the number of leading 1s in $\mathrm{rref}(A)$ and the nullity of A is the number of free variables.