Homework 8: Basis

This homework is due on Friday, February 19, respectively on Thursday, February 18, 2016.

1 Which of following sets are linear spaces? Check in each case the three properties characterizing a linear space.
a) $W = \{(x, y, z) \mid x + y + z = 2\}$
b) $W = \{(x, y, z, w) \mid x = w, y = z\}$
c) $W = \{(x, y, z) \mid x = y = z\}$
d) $W = \{(t + 1, t + 2, t + 3) \mid t \in \mathbf{R}\}$
e) $W = \{(x, y, z) \mid x^2 + y^2 = 0\}$
f) $W = \{(x, y, z) \mid x^2 = 0\}$

2 Let V, W be linear subspaces of R^3. Which are linear spaces?
a) (2 points) The union $V \cup W$ of V and W.
b) (2 points) The intersection $V \cap W$ of V and W.
c) (2 points) The set $V^{\perp}$ of vectors perpendicular to V.
d) (2 points) the intersection $V \cap W$ of V and unit sphere W : $x^2 + y^2 + z^2 = 1$.
e) (2 points) the "augmented difference" $V \setminus W \cup \{(0, 0, 0)\}$ of V, W, where $V \setminus W$ consists of all v which are in V but not in W.

3 Check whether the given set of vectors is linearly independent

a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} \right\}$. b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$. c) $\left\{ \begin{bmatrix} 20 \\ 16 \end{bmatrix}, \begin{bmatrix} 2 \\ 18 \end{bmatrix}, \begin{bmatrix} 2 \\ 19 \end{bmatrix} \right\}$.

4 Find a basis for the image as well as as a basis for the kernel of the following matrices

a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

5 The orthogonal complement of a subspace V of R^n is the set $V^\perp$ of all vectors in R^n that are perpendicular to every single vector in V. Find a basis for the orthogonal complement in each case:
a) The line L in R^5 spanned by $\begin{bmatrix} 1 & 2 & 2 & 1 & 1 \end{bmatrix}^T$, (If v is a row vector v^T denotes the corresponding column vector).
b) The plane Σ in R^4 spanned by $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^T$.
c) The space $V = \{(0,0)\}$ in the two-dimensional plane R^2.

Basis

V **linear space** if $0 \in V$, $v + w$ is in V for $v, w \in V$ and λv is in V for $v \in V$ and $\lambda \in \mathbf{R}$. Examples: $V = \ker(A)$ or $V = \text{im}(A)$. If V, W are linear spaces and $V \subset W$, then V is a **subspace** of W. A set $\mathcal{B}$ of vectors $\{v_1, \ldots, v_n\}$ **spans** V if every $v \in V$ is a sum of vectors in $\mathcal{B}$. It is linear independent if $a_1 v_1 + \cdots + a_n v_n = 0$ implies $a_1 = \ldots = a_n = 0$. It is a **basis** of V if it both **spans** V and is linearly independent. Example: $\{e_1, \ldots, e_n\}$ is a basis of R^n. Use row reduction to check whether vectors $\mathcal{B} \subset \mathcal{R}^\backslash$ are a basis in R^n: place $v_i \in \mathcal{B}$ as columns in a matrix A, then row reduce A. It is a basis if it row reduces to I_n. If every row will has a leading 1, then $\mathcal{B}$ spans R^n: the image of A is R^n. If every column has a leading 1, then $\mathcal{B}$ are linearly independent: the kernel of A is $\{0\}$.