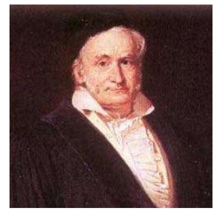
Some Notes on complex numbers Harvard University, O.Knill, 2004

"The shortest path between two truths in the real domain passes through the complex domain."
Jacques Hadamard (1865-1963)

THE SYMBOL I. Introducing the symbol $i = \sqrt{-1}$ and extending all usual calculation rules using $i^2 = -1$ leads to the algebra of complex numbers $z = a + ib$. The number $z = 17 - 12i$ is an example of a complex number. Real numbers like $z = 3.2$ are considered complex numbers too. The mathematican Johann Carl Friedrich Gauss (1777-1855) was one of the first to use complex numbers seriously in his research. Still, as late as 1825 he claimed that "the true metaphysics of the square root of -1 is elusive".

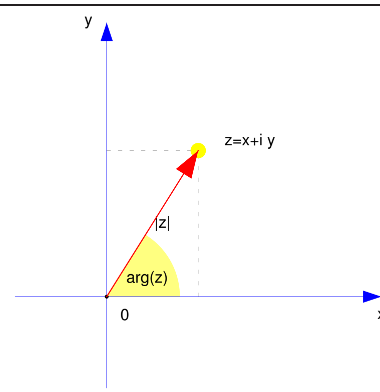


EULER FORMULA. The identity

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}$$

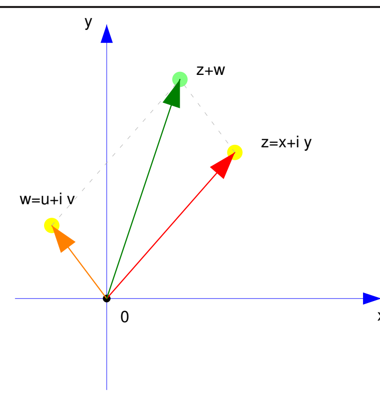
It can be proven using the power series $\cos(x) = 1 - x^2/2! + x^4/4! - ..$, $\sin(x) = x - x^3/3! + x^5/5! ...$ and $e^x = 1 + x + x^2/2! + x^3/3! + ...$

POLAR REPRESENTATION. Because complex numbers $z = x + iy$ can be realized as vectors (x, y) in the plane, we can represent them in polar coordinates $z = x + iy = r\cos(\theta) + ir\sin(\theta)$. Euler's formula gives $z = re^{i\theta}$. The plane is also called the **complex plane** or the **Gauss plane**.

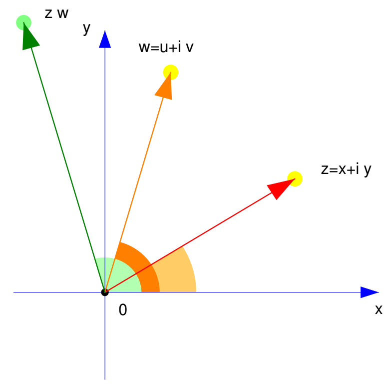


ADDITION. $z = x + iy, w = u + iv$ $z + w = (x + u) + i(u + v)$. Adding $-w = u + iv$ to z is called subtraction and denoted by $z - w$. In the Gauss plane, addition can be done by drawing the parallelogram spanned by the vectors (x, y) and (u, v) to get the vector $(x + u, y + v)$.

Examples: $(5 + 7i) + (3 - 4i) = 2 + 3i$.
$(3 + i) - (2 + i) = 1$.



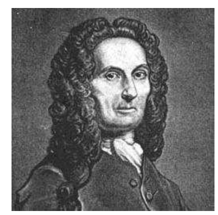
MULTIPLICATION. With $z = x + iy$ and $w = u + iv$ define $zw = (xu - yv) + i(xv - uy)$. Because multiplying $z = re^{i\theta}$ and with $w = se^{i\phi}$ gives $zw = rse^{i(\theta+\phi)}$, we see that the length of the product $|zw|$ is the product of the lengths $|z||w|$ of the z and w and that the polar angle $\theta + \phi$ of zw is the sum of the polar angles θ and ϕ of z and w. You have verified this in a homework by realizing complex numbers as rotation dilation matrices.



Examples: $(3 + 2i)(1 - i) = 5 - i$.
$(1 + i)^2 = 2i$.

DE MOIVRE MAGIC. With $z = e^{i\theta}$, we have $z^n = e^{in\theta}$ and so

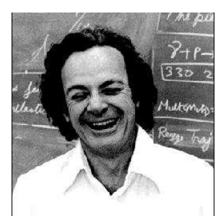
$$(\cos(\theta) + i\sin(\theta))^n = e^{in\theta}$$



Writing out the real and imaginary part leads to interesting identities. For example, for $n = 3$, we get $\cos(3\theta)$ + i $\sin(3\theta)$ = $(\cos(\theta) + i\sin(\theta))^3$ = $\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$ + i $(3\cos^2(\theta)\sin(\theta) - \sin^3(\theta))$. Comparing real and imaginary parts gives identities which would be harder to derive without this magical stunt.

AN AMAZING FORMULA.

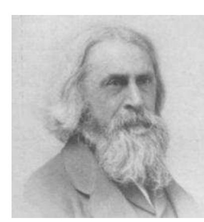
$$1 + e^{i\pi} = 0$$



combines the constants $0, 1, e, \pi$ in a magical way. It is considered one of the 5 most beautiful formulas in mathematics. Richard Feynmann called it as a 15 year old

"the most remarkable formula in math".

In the book of E. Kasner and J. Newman, "Mathematics and the Imagination", Benjamin Peirce is quoted after proving this formula here in front of a Harvard class:

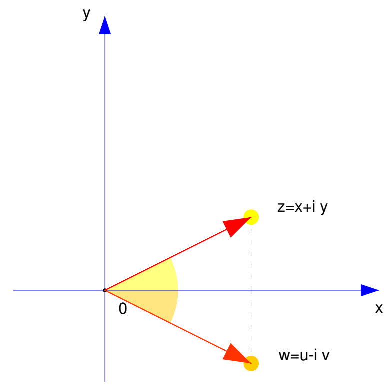


Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore, we know it is the truth.

We can only repeat that statement modifying the start of the sentence to "Ladies and Gentlemen" of course.

COMPLEX CONJUGATE.

The **complex conjugate** of $z = x + iy$ is $\overline{z} = x - iy$. Example: $\overline{3 + 4i} = 3 - 4i$.

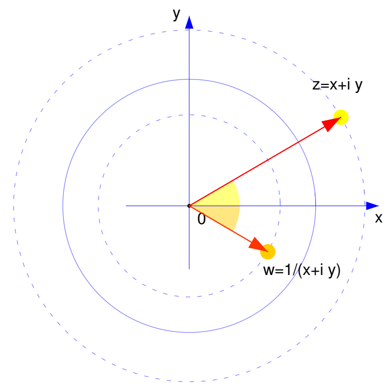


Example: $\overline{3 + 6i} = 3 - 6i$.

ABSOLUTE VALUE $|z|$ of a complex number $z = x+iy$ is $\sqrt{x^2 + y^2}$. We can also write $|z|^2 = z\overline{z}$. The absolute value is also called the **modulus**.

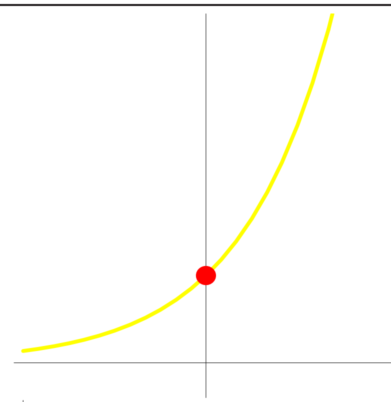
Example: $|1 + 2i| = \sqrt{1^2 + 2^2} = \sqrt{5}$.

DIVISION. With $z = x + iy$ and $w = u + iv$, we have $z/w = z\overline{w}/w\overline{w} = z\overline{w}/|w|^2 = (xu + yv)/(u^2 + v^2) + i(-xv - uy)/(u^2 + v^2)$.

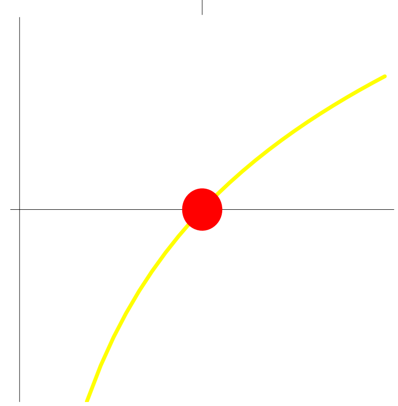


Examples: $1/i = -i$.
$1/(1 + i) = (1 - i)/\sqrt{2}$.

REAL EXP.
The graph of the real function exp is monotone and above the x axes. Because $\exp'(x) = \exp(x)$ the slope of the graph grows exponentially too.

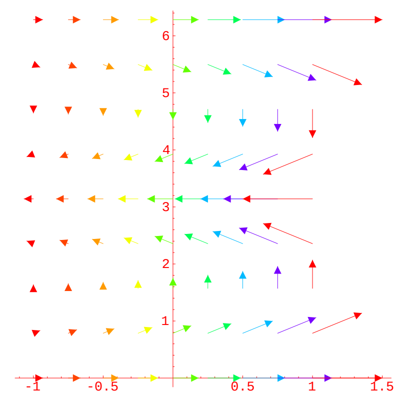


REAL LOG.
The graph of the real function log is monotone too and defined only on the positive x-axes. Since $\log'(x) = 1/x$ the slope of the graph becomes smaller and smaller for larger x.



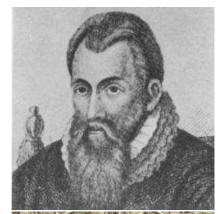
EXP. We can extend the definition e^z to any complex number z with $e^{x+iy} = e^x e^{iy} = e^x \cos(y) + e^x \sin(y) i$.

Understanding the "graph" of the function "exp" is not so easy because we would have to plot for each $z = x+iy$ the result $\exp(z) = u + iv$. In multi-variable calculus, you will learn to work with functions of several variables. A possibility to visualize "exp" is to draw the vector (u, v) at the point (x, y).



Examples: $e^{1+i\pi} = ee^{i\pi} = -e$.
$e^{2-3i} = e^2 e^{-i} = e^2 \cos(3) - ie^2 \sin(3)$.

LOG. A value $\log(z)$ can be defined for any complex number z which is different from 0. This is done by $\log(z) = \log|z| + i\arg(z)$, where $\arg(z) = \theta$ is the angle in $[0, 2\pi)$ when writing $z = re^{i\theta}$. Using the polar representation, you can verify that $\exp(\log(z)) = z$ and $\log\exp(z) = z$. Jost Bürgi (1552-1632) developed logarithms independently of John Napier (1550-1617). While Napier's approach was algebraic, Burgi's point of view was geometric. It is believed that Bürgi created a table of logarithms before Napier by several years, but did not publish it until later. Johann Kepler who admired Bürgi as a mathematician states in the introduction to his Rudolphine Tables (1627): "... the accents in calculation led Justus Byrgius (Jost Bürgi) on the way to these very logarithms many years before Napier's system appeared; but being an indolent man, and very uncommunicative, instead of rearing up his child for the public benefit he deserted it in the birth."



Examples: $\log(i) = i\pi/2$
$\log(3+4i) = 5 + \arctan(4/3)i$.

ARBITRARY EXPONENTIALS. Using the log, we can define z^w for two complex numbers z, w by $z^w = e^{w\log(z)}$.

Example: $(1+i)^{2+i} = e^{(2+i)\log(1+i)} = e^{(2+i)\sqrt{2}\pi/4} = e^{2\sqrt{2}\pi/4}(\cos(\sqrt{2}\pi/4) + i\sin(\sqrt{2}\pi/4))$.

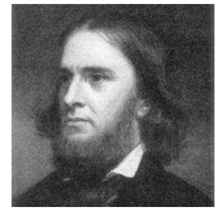
EXP AND LOG RULES. The usual rules for exp and log carry over to the complex:

- $\exp(z+w) = \exp(z)\exp(w)$
- $(e^z)^w = e^{zw}$.
- $\log(zw) = \log(z)\log(w)$
- $\log(z^w) = w\log(z)$

Examples: $(e^{1+i} \cdot e^{1-i})^{2-i} = (e^2)^{(2-i)} = e^{(4-2i)} = e^4 e^{-2i} = e^4 \cos(2) - ie^4 \sin(2)$.
$\log((3+4i)^{(1-i)}) = (1-i)\log(3+4i) = (1-i)(5 + i\arctan(\frac{4}{3}))$.

MORE EXAMPLES AND A MYSTERIOUS FORMULA.

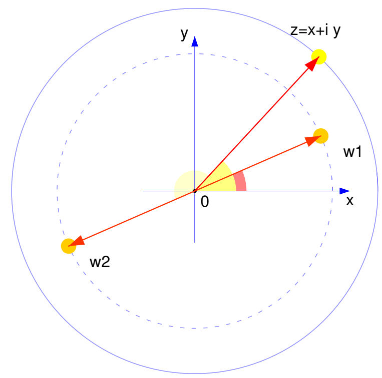
- $\log(i) = \log|i| + i\arg(i) = i\pi/2$.
- $i^i = e^{i\log(i)} = e^{i(i\pi/2)} = e^{-\pi/2}$.
- $(-1)^i = e^{-\pi}$.



The last two examples are remarkable! The second formula implies $\boxed{\pi = \log(i)\frac{2}{i}}$. To cite Harvards Benjamin Peirce (1809-1880) again: he called it a **"mysterious formula"**.

By the way: the third formula can be used to show that $e^{-\pi}$ is **transcendental**, which means that it is not the root of a polynomial with integer coefficients, a problem posed by David Hilbert in 1900 which he thought to be more difficult than the Riemann Hypothesis but which was solved by the Russian mathematician Gelfond in 1929.

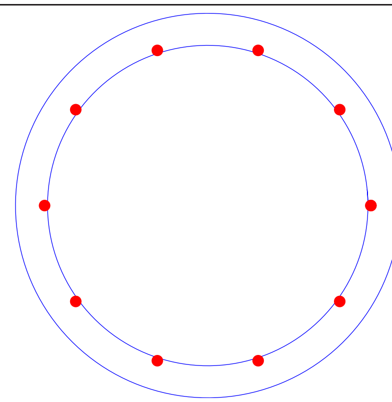
SQUARE ROOT. $z = re^{i\theta}$ has the square root $w = \sqrt{r}e^{i\theta/2}$. But since $z = re^{2\pi i\theta}$ also, we have an other root $\sqrt{r}e^{i\theta/2+\pi} = -w$. Indeed both w and $-w$ satisfy $w^2 = -1$. We see that any complex number different from 0 has exactly two square roots and that the sum of these two roots is zero.



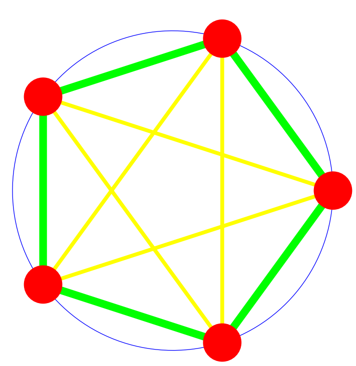
EXAMPLES. For $z = 3+4i$, one has $z = 5e^{i\phi}$ with $\phi = \arctan(4/3)$ and so $\sqrt{z} = \pm\sqrt{5}e^{i\phi/2}$.

N-TH ROOTS. $z^n = w = r^{i\theta}$ has the solutions $z = w^{1/n} = r^{1/n}e^{i\theta/n+k2\pi/n}$. All these solutions are located on a circle of radius $|z|^{1/n}$.

Examples: The fourth roots of 1 are the complex numbers $1, i, -1, -i$.
The third roots of $8i$ are the complex numbers $2e^{i\pi/12}, 2e^{i5\pi/12}, 2e^{i9\pi/12}$.



EXAMPLE. For $z = -i$, find all the 5'th roots. The solutions are $z_k = e^{-i\pi/2+k\pi/5}$, where $k = 0,1,2,3,4$. The points are located on a regular pentagon. The length of a side is $|z_1 - z_2|$. The length of a diagonal is $|z_1 - z_3|$. We can compute that the ratio of the diagonal length and the side length is the **golden ratio** $(\sqrt{5}+1)/2$, one of the most remarkable numbers in mathematics. This is why the pentagon often appears in "magic". An example is the book "The amulet of Samarkand" by Jonathan Stroud, I just read.



TRIGONOMETRIC IDENTITIES. More trigonometric identities can be derived similarly to the de Moivre magic: the Euler formula $e^{ix} = \cos(x) + i\sin(x)$ implies

$$\cos(x+y) + i\sin(x+y) = e^{i(x+y)} = e^{ix}e^{iy} = (\cos(x) + i\sin(x))(\cos(y) + i\sin(y))$$

which leads to **trigonometric identities** by comparing real and imaginary part

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y)$$

In the special case $x = y$, one gets the important identities

$$\cos(2x) = \cos^2(x) - \sin^2(x), \sin(2x) = 2\sin(x)\cos(x) .$$

By adding up identities for $x+y$ and $x-y$, we get

$$\cos(x-y) + \cos(x+y) = 2\cos(x)\cos(y), \sin(x+y) + \sin(x-y) = 2\sin(x)\cos(y)$$

Because $\cos(x) = \sin(x + \pi/2)$, we get also

$$\cos(x-y) - \cos(x+y) = 2\sin(x)\sin(y) .$$

All together, we have the useful multiplication identities

$$\sin(x)\sin(y) = \frac{\cos(x-y)-\cos(x+y)}{2}$$

$$\cos(x)\cos(y) = \frac{\cos(x-y)+\cos(x+y)}{2}$$

$$\sin(x)\cos(y) = \frac{\sin(x+y)+\sin(x-y)}{2}$$

WHERE ARE COMPLEX NUMBERS USED?

- Mechanics: like describing epicycles: $e^{it} + e^{ikt}$.
- Fourier series: simplifications.
- Geometry: i.e. find the length of a diagonal in a pentagon.
- Quantum mechanics: wave functions are complex valued, path integrals using imaginary time.
- Integration like $\int \sin^2(x)dx = \int (e^{ix} - e^{-ix})^2/(2i)^2 dx$
- Simplifying trigonometry
- Linear algebra: linearization.
- Differential equations appearing in electrotechnics
- Statistics: tool to compute moments like variance
- Particle physics: symmetry groups are complex matrices
- Finding real integrals.

BEYOND COMPLEX NUMBERS. To define subtraction for arbitrary natural numbers, one has to include **negative numbers**. To make division possible for any two nonzero integers, **rational numbers** are introduced. In order to take limits, one has to include also irrational numbers leading to **real numbers**. In order that any polynomial has roots, we used **complex numbers**. Does one have to go further? Can one? Does one want to? Yes, there are extensions to the complex numbers but things become somehow more unpleasant. The only extensions possible are the **quaternions** and the **octonions**. But there is a prize to be payed: the multiplication of quaternions is no more commutative, the multiplication of octonions is even no more associative: $a(bc) \neq a(bc)$ in general.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

The discovery of quaternions is attributed to William Rowan Hamilton. Conway and Guy write in their book: 'Every morning, on coming down to breakfast, his young son would ask him, "Well, Papa, can you multiply triplets?" but for a long time he was forced to reply with a sad shake of his head, "No, I can only add and subtract them".' When Hamilton finally succeeded, he cut with a knife on a stone of Brougham bridge the fundamental formula with the symbols

$$i, j, k : i^2 = j^2 = k^2 = ijk = -1.$$

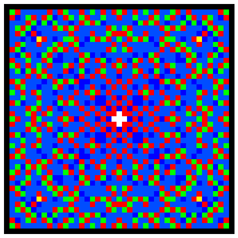
The natural numbers are the most complex numbers.
The complex numbers are the most natural numbers.

THE FUNDAMENTAL THEOREM OF ALGEBRA.

A polynomial of of degree n has exactly n roots in the complex.

Remark. Gauss was the first to give a proof in his theses. At first, before writing his dissertation, Gauss still believed that there would be a hierarchy of complex numbers. He thought that whenever a polynomial can not be factored, one would have to extend the number system.

GAUSSIAN INTEGERS. Complex numbers $z = a + ib$ with integer a, b are called **Gaussian integers**. In the same way as usual integers can be decomposed into **primes**, one can also decompose Gaussin integers into **Gaussian primes**. While some regular primes are prime also as Gaussian primes like 3, there are regular primes like 5 which are no more prime as Gaussian primes: we have a factorization $5 = (1 + 2i)(1 - 2i)$. To the right we colored each complex integer according to the number of factors.



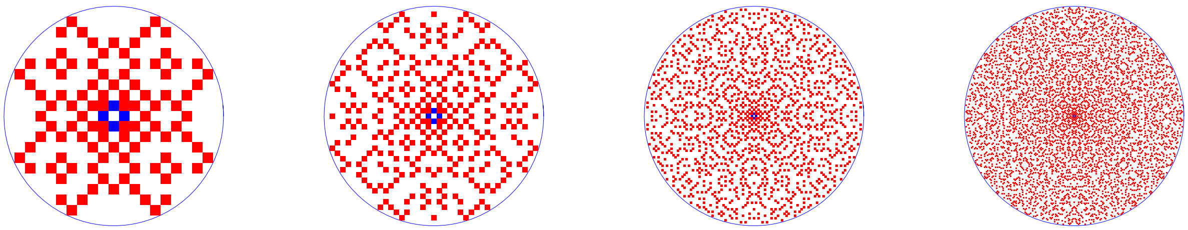
Below are the Gaussian primes displayed in the complex plane. Mathematica has already built in the feature to check for Gaussian Primes like

```
PrimeQ[2 + I, GaussianIntegers-> True].
```

Factorization can be done with

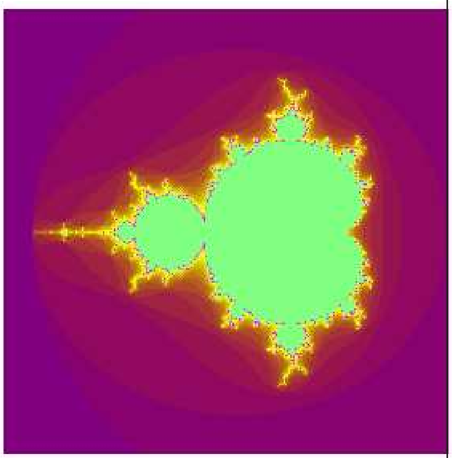
```
FactorInteger[n, GaussianIntegers-> True].
```

Gaussian integers are factored in a similar way then usual integers. The **baby algorithm** just tries division with integers of smaller modulus. If $z = z_1 z_2$ is a factorization, then $|z|^2 = |z_1|^2 + |z_2|^2$ so that $x^2 + y^2 = (x_1^2 + y_1^2)(x_2^2 + y_2)^2$.



MANDELBROT SET The **quadratic map**

$$f_c : z \mapsto z^2 + c$$



with a complex parameter c can be iterated: the game is to start with a complex number z like 0 and applying the rule f_c on it. Let $c = 1$ for example. We get a sequence of complex numbers $0 \to i \to i^2 + i = i - 1 \to (i - 1)^2 + i \to ....$ In the parameter space C there is a **Mandelbrot set** M, which is defined to be the set of parameters for which the orbit $0 \to c \to c^2 + c...$ stays bounded. You see for example that the point $z = 2$ is not in M since the sequence of numbers $f_c(z) = 6, f_c(z) = 42$ escapes to infinity. The parameter point $c = 0$ however is a point in M.

COMPUTING THE MANDELBROT SET

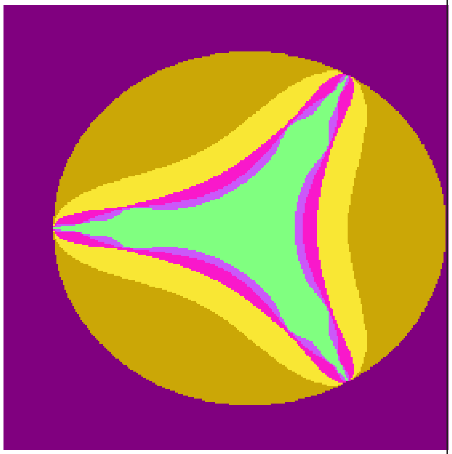
The Mandelbrot set is a set in the parameter domain. In order to see, who we color a point c in the complex plane. We iterate T_c starting at $z = 0$ and look how long it takes to have a modulus larger than some value like 2. If we are close to the set, it will take long to get there, if we are inside the Mandelbrot set, we will never get there.

```
M = Compile[{x, y}, Module[{z = x + Iy, k = 0}, While[Abs[z] < 2.&&k < 50, z = z^2 + x + Iy; + + k]; k]];
DensityPlot[50 − M[x, y], {x, −2.2, 1.}, {y, −1.6, 1.6}, PlotPoints− > 500, Mesh− > False]
```

MANDELBAR SET. The same construction can be done by replacing the quadratic map with the **conjugate quadratic map**

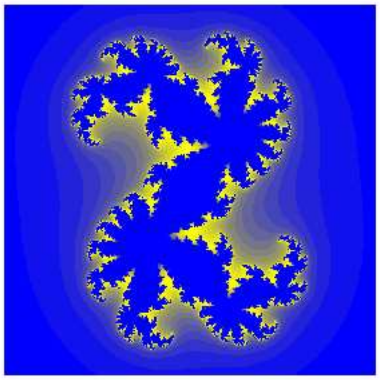
$$f_c : z \mapsto \overline{z}^2 + c \, .$$

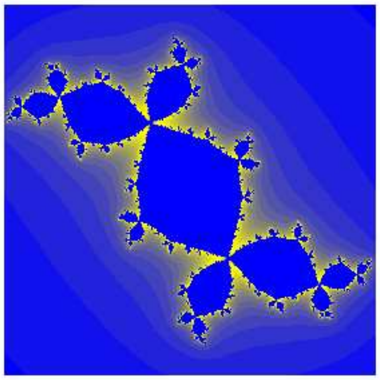
The corresponding set is called the **Mandelbar set**.
People are interested in this set because its topological properties are different from the one believed to be true for the Mandelbrot set. This is out of the scope but the last big open problem in complex dynamics: one believes that the Mandelbrot set is locally connected meaning that every neighborhood of a point contains a connected open neighborhood. The mandelbar set does not have this property. There are points, where the set resembles the graph of the function $f(x) = \sin(1/x)$ which is not locally connected at $(0, 0)$.



JULIA AND FATOU SETS. The map f_c leaves a set $J_c \subset C$ called **Julia set** and its complement F_c the **Fatou set** invariant.
The Julia set is in general a fractal and at in a complicated way mixed with the Fatou set. It is ironic that Gaston Julia (1893-1978) and Pierre Fatou (1879-1929) were not very well spoken on each other. They competed both for the 1918 'grand priz' of the academie of sciences and produced similar results leading to a priority dispute. Julia was wounded in world war I. He lost his nose and had to wear a leather strap across his face for the rest of his life. He carried on his mathematical researches in hospital.

EXAMPLES OF JULIA SETS.





"The dragon" c=0.36+0.1 i "The cauliflower" c=0.25 "The Douady rabbit" c=-0.121 + 0.739 i

Julia sets are in general **fractals** meaning that their dimension is between 1 and 2. By the way, one knows that the boundary of the Mandelbrot set has dimension 2. It is a very complicated

COMPUTING JULIA SETS. Pictures of Julia sets J_c to a parameter c can be computed in a similar way as the Mandelbrot set. Start with a point z and iterate it and look how long it takes to get outside a certain disk. Points on the Julia set will never escape and if one is close to the set, the escape time will be long. The next two lines generate the Douady Rabbit in Mathematica.

```
J = Compile[{x, y, u, v}, Module[{z = u + Iv, k = 0}, While[Abs[z] < 200.&&k < 50, z = z^2 + x + Iy; + + k]; k]];
DensityPlot[50 − J[−0.121, 0.739, u, v], {u, −1.3, 1.3}, {v, −1.3, 1.3}, PlotPoints− > 500, Mesh− > False];
```

COMPLEX NUMBERS IN REAL LIFE. Where will you encounter complex numbers most likely?

- If you take a linear algebra course, you will also be exposed to Fourier theory. It turns out that Fourier theory is much more elegant in the complex setup. A function $f(x)$ can be written as a sum $f(x) = \sum_n a_n e^{\pi n x}$ called **Fourier series**.

- In **linear algebra**, eigenvalues can become complex even for real matrices A. For example, a rotation by an angle α in the plane has the eigenvalues $e^{i\alpha}$ and $e^{-i\alpha}$.

- If you study **differential equations**, stability issues are related to complex numbers. For example, for the differential equation $y''(x) = -ay(x)$ the numbers $\lambda = \pm\sqrt{-a}$ are relevant.

- In **statistics** the tool of characteristic functions is used. If X is a random variable, then $\chi_X(t) = \mathrm{E}[e^{itX}]$ is called the characteristic function of X, where $\mathrm{E}[f(x)]$ is the expectation of the random variable $f(X)$. One of the main handy things about this functions is that if X, Y are independent, then $\chi_{X+Y}(t) = \chi_X(t)\chi_Y(t)$.

- If you use a **computer algebra system** and you ask to find all the roots of a polynomial, then the software will give you back the real as well as the non real solutions. Due to rounding errors, it can happen that you obtain non-real solutions even if the solutions are real.

LITERATURE.

- Currently the best historically introduction to complex numbers is the book by Paul Nahin: "An imaginary tale: the story of $\sqrt{-1}$". The Hadamard citation at the beginning of this handout is taken from there.
- The marvelous booklet of Conway and Guy "The book of Numbers" contains an introduction to complex numbers and Gaussian primes. The style of this book is hard to beat: densely packed with information, no unnecessary talk, but still readable like a novel. Its probably the most beautiful book on numbers.
- A nice introduction to Gaussian integers is in Appendix I of the gem "The geometry of numbers" by C.D. Olds, Anneli Lax and Giuliana Davidoff.
- The bestseller "Gamma: Exploring Euler's constant" has also a introduction into complex analysis and provides also some good history background on (complex) logarithms.
- We cited from the book "Mathematics and the Imagination" of Kasner and Newman, which is available in the 'budget friendly' but nevertheless excellent Dover series.
- Most calculus textbooks contain some introduction to complex numbers. Complex logarithms are often neglected in calculus textbooks.
- As a survival or refresher guide, SparkNote on Complex Numbers, by Kenny Shirley (http://www.sparknotes.com/math/precalc/complexnumbers) come handy.
- An inspiring book in the spirit of experimental mathematics is "Computational Number theory" by David Bressoud and Stan Wagon, where in some chapter the reader is also invited to do some experiments with Gaussian integers.
- There are many online resources on complex numbers. The photo illustrations of some Mathematicians displayed in this handout were taken from the "Mac Tutor History of Mathematics Archive" at the School of Mathematics and Statistics at the University of St Andrews. Also the citation of Gauss of 1825, that the "true metaphysics of the square root of -1 is elusive" is from that article.
- One picture of Benjamin Peirce was taken from: Julian Lowell Coolidge, "The Development of Harvard University, 1869-1929, Since the inauguration of President Eliot, 1869-1929 Chapter XV. Mathematics, 1870-1929".
- The book "The golden Ratio, the Story of Phi, the Word's Most Astonishing Number" by Mario Livio is a good source on the golden number. A personal side remark: Livio, a leading astronomer, was the PhD advisor of my wife.
- The topic of complex iteration has been treated extensively in the literature for a general audience. There are dozens of entertaining books on the subject. The book of James Gleick, "Chaos" made it into a natural bestseller. The book "The Beauty of Fractals" of H-O. Peitgen and P.H Richter of 1986 still contains some of the most beautiful pictures of the Mandelbrot set. The pictures here were produced either with Mathematica or the software "xfractint" from the "stone soupe group".
- An affordable and rather mathematical booklet has been authored by by Lennart Carleson and Theodore W. Gamelin. Its title is "Complex Dynamics".

PROBLEMS.

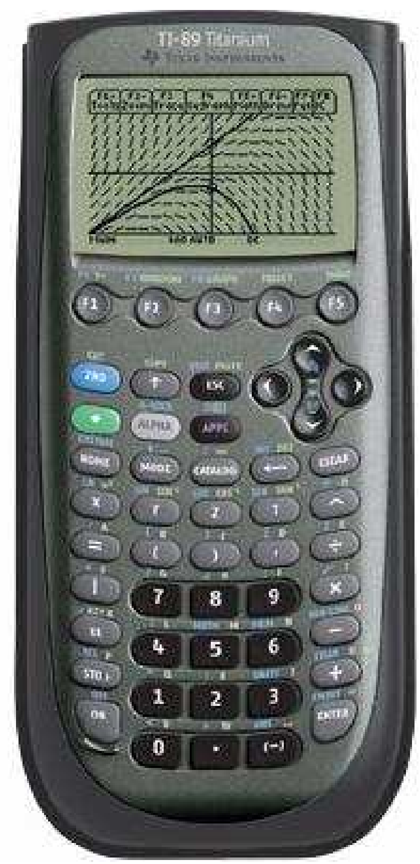
1) What is $(1+3i)/(4+i)$?
2) What is i^{100002}?
3) Find all the roots of x^2+3x+3.
4) Find all the solutions of $x^2+9=0$.
5) Verify that $\sin(ix) = i\sinh(x)$, where $\sinh(x) = (e^x - e^{-x})/2$.
6) Find all the third roots of $3+4i$.
7) Find $\log(1+\sqrt{3}i)$.
8) Derive $2\sin(x)\cos(x) = \sin(2x), \cos^2(x) - \sin^2(x) = \cos(2x)$ from de Moivre.
9) You iterate the map $f(z) = z^2 + i$. What is $f(f(f(f(z))))$ for $z = 1$?
10) Find the prime factorization of the Gaussian integer $z = 5 + 7I$.

APPENDIX: COMPLEX NUMBERS AND GRAPHICS CALCULATORS.

Pocket Calculators: While smaller pocket calculators like the TI-30 give an error message when trying to compute $\sqrt{-1}$, more advanced calculators like the TI-89 know the arithmetic of complex numbers. The symbol i is accessible as

< 2nd >< catalog >

(you see the i symbol above the < catalog > key). Note that graphics calculators should be considered rather a toy. They are not very appropriate on the college level. The graphic abilities of calculators like TI-89 Titanium are very limited. There are many reasons to learn a real computer algebra system (CAS).



Advantages of graphics calculators: always on and ready, cheap (50-150 dollars). Often familiar to students from high school.

Disadvantages of graphics calculators: small memory, poor graphics, complicated interface, difficult to connect with other programs or other computers. Soon obsolete.

APPENDIX: COMPLEX NUMBERS AND COMPUTER ALGEBRA SYSTEMS.

Computer algebra systems: Mathematica and Maple are examples of computer algebra systems (CAS).

Mathematica:

- Solve[$x^3 + 5x + 10 == 0, x$]
- $(2 + 3 * I)/(3 + 7 * I)$
- Log[I]
- Sqrt[-2]
- Exp[Pi * I]
- Conjugate[1 + I]

Maple:

- solve($x^3 + 5x + 10 = 0$);
- $(2 + 3 * I)/(3 + 7 * I)$;
- log(I);
- sqrt(-1);
- exp(Pi * I);
- conjugate(1 + I);

(Note that maple needs a semicolon after each command, a semicolon is also possible in Mathematica, but then Mathematica does not display the result) Using a different CAS is like driving an other car, a rough translation of Mathematica to maple for example is to replace uppercase to lower case, square brackets with round brackets and adding a semicolon at the end.

APPENDIX: SOLUTIONS TO PROBLEMS.

1) $(1 + 3i)/(4 + i) = (1 + 3i)(4 - i)/\sqrt{17} = (7 + 11i)/\sqrt{17}$
2) Note that $i^4 = 1$ so that $i^{1000} = 1$ too and the answer is -1.
3) $(-3 \pm i\sqrt{3})/2$.
4) $\sqrt{-9} = 3\sqrt{-1} = \pm 3i$.
5) $\sin(ix) = (e^{iix} - e^{-iix})/i = i(e^{i} - e^{-x})/2 = i\sinh(x)$.
6) $3 + 4i = 5e^{i0}$ so that $(3 + 4i)^{(1/3)} = 5^{1/3}e^{2\pi ki/3}$,$k = 0, 1, 2$.
7) $z = 1 + \sqrt{3}i = 2e^{i\pi/3}$ so that $\log(z) = \log(2) + i\pi/3$.
8) $2\sin(x)\cos(x) = \sin(2x) = \mathrm{Im}(e^{2ix})$, $\cos^2(x) - \sin^2(x) = \mathrm{Re}(e^{i2ix}) = \mathrm{Re}(e^{ix})^2\cos(2x)$.
9) $80 - 17i$.
10) $25 + 49 = 74 = 2 \cdot 37$. We have to look for Gaussian integers p, q with $|p|^2 = 2$ and $|q|^2 = 37$. Indeed $(1 + i)(6 + i)$ is the factorization.

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