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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

1) T F If A is the matrix representing a shear in the plane then $\det(A - I_2) = 0$.

Solution:
A shear has an eigenvalue 1.

2) T F Two matrices are similar if and only if they have the same eigenvalues.

Solution:
The shear A and the identity $B = 1$ is a counter example. They have the same eigenvalues but they are not similar.

3) T F Every orthogonal $n \times n$ matrix satisfies $A^2 = I_n$.

Solution:
The definition is $A^T A = I_n$. There are some orthogonal transformations which satisfy this. They are reflections. But not all of them have this relation.

4) T F The eigenvalues of a $n \times n$ matrix A do not change under row reduction.

Solution:
A scaling factor already changes the eigenvalues. Start with the matrix $2I_2$ and row reduce. We end up with I_2. The eigenvalues of the initial matrix are 2 which is different from the eigenvalues of the later.

5) T F Every 2×2 rotation matrix can be diagonalized over the complex numbers.

Solution:
There can be complex eigenvalues.

6) T F The recursion $x(t+1) = x(t)^2 - x(t-1)^2$ can be written as a vector equation $\vec{v}(t+1) = A\vec{v}(t)$ for a 2×2 matrix A and a vector $\vec{v}$.

Solution:
It is not linear.

7) T **F** The product of two rotations in the plane can be a reflection at a line.

Solution:
Rotations have determinant 1, a reflection has determinant -1.

8) **T** F The rank of an $n \times n$ matrix A is the same as the rank of A^T.

Solution:
There are the same number of leading 1.

9) **T** F For any 3×3 matrix, we have $\det(A^3/3) = \det(A)^3/27$.

Solution:
We have seen that $\det(\lambda A) = \lambda^n \det(A)$.

10) T **F** A discrete dynamical system is asymptotically stable if the absolute value of the trace A and determinant are both smaller than 1.

Solution:
The eigenvalues, not the trace.

11) **T** F A reflection dilation in the plane has zero trace.

Solution:
The diagonal entries have different sign.

12) **T** F The matrix $A = \begin{bmatrix} 7 & 0 \\ 5 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 3 & 0 \\ 5 & 7 \end{bmatrix}$.

Solution:
Because both have the same trace and determinant, their eigenvalues are the same. Because the eigenvalues are different they can both be diagonalized to the same diagonal matrix.

13) | T | F | If two 4×4 matrices have each the eigenvalues 1 with algebraic multiplicity 4 and the same geometric multiplicity 2, then they are similar.

Solution:
A counter example was in the homework

14) | T | F | If A has only the trivial kernel so that $A^T A$ is invertible then the least square solution of $Ax = b$ is unique.

Solution:
Yes, then the quadratic error is zero.

15) | T | F | If $|A^{-n}\vec{v}|$ goes to infinity for $n \to +\infty$ for every non-zero $\vec{v}$, then $|A^n\vec{v}|$ goes to zero for $n \to +\infty$ for every $\vec{v}$.

Solution:
A shear is a counter example.

16) | T | F | There is an orthogonal anti-symmetric matrix. (Remember that anti-symmetric=skew symmetric means $A^T = -A$).

Solution:
The rotation by $\pi/2$ is an example.

17) | T | F | If $A = QR$ is the QR decomposition of a square matrix, then A is stable if and only if R is stable.

Solution:
Let Q be the rotation by 90 and $R = Diag(2, 1/3)$.

18) T **F** For every 2×2 matrix A, we know that A is similar to A^{-1}.

Solution:
The eigenvalues the geometric multiplicities are the same.

19) **T** F If an invertible A is similar to B then $A^{10} + A^{-1}$ is similar to $B^{10} + B^{-1}$.

Solution:
$S^{-1}AS = B$ can be inverted to get $S^{-1}S^{-1}S = B^{-1}$.

20) **T** F If A is a symmetric 2×2 matrix, has trace 2 and determinant 1, then it is the identity.

Solution:
It is conjugated to the identity and so the identity.

	Total

Problem 2) (10 points) No justifications are needed.

a) (2 points) Which of the following complex numbers are real?

Number	is real	is not real
$z = e^{i\pi}$		
$z = 1/i$		
$z = i^4 - i^2$		
$z = e^{i\pi/2} - i$		

Solution:
Only $1/i$ is not real. Note that $e^{i\pi} = -1$ and $i^2 = -1$ and $e^{i\pi/2} = i$.

b) (2 points) Which matrices have the property that the system $x(t+1) = Ax(t)$ is stable?

Matrix	stable	not stable
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		
$\begin{bmatrix} 0.8 & 1 \\ 0 & -0.7 \end{bmatrix}$		
$\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$		
$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$		

Solution:
Only the identity matrix is not stable. For the others, the eigenvalues satisfy $|\lambda_j| < 1$.

c) (2 points) Which identities do always hold?

Identity	always true	not always true
$\det(A^2 - B^2) = \det(A^2) - \det(B^2)$		
$\det(A^2 - B^2) = \det(A - B)\det(A + B)$		
$\mathrm{tr}(A^2 - B^2) = \mathrm{tr}(A^2) - \mathrm{tr}(B^2)$		
$\mathrm{tr}(A^2 - B^2) = \mathrm{tr}(A - B)\mathrm{tr}(A + B)$		

Solution:
Only the third identity is always true. For the second property, we would need that A, B commute.

d) (2 points) Which of the following formulas define the projection onto the linear space $V = \text{im}(A)$, if A has no kernel?

Formula	is a projection onto V	is not a projection onto V
AA^T		
$A(A^TA)^{-1}A^T$		
$(A^TA)^{-1}A^T$		
$(AA^T)^{-1}A^T$		

Solution:
Only the second formula is the projection. The first one only applies if A has orthonormal column vectors.

e) (2 points) Assume S is a matrix which contains an eigenbasis of an $n \times n$ matrix A as columns. Which of the following statements are true?

Statement	is always true	is not always true
SAS^{-1} is diagonal		
$S^{-1}AS$ is diagonal		
A is diagonal		
S is diagonal		

Solution:
Only the second one is always true.

Problem 3) (10 points) No justifications are needed

a) (2 points) In the following, A is a 2×2 matrix of a given type.

Type	$\det(A) = 1$	A^2 is same Type	always real eigenvalues	diagonalizable
Rotation				
Projection				
Symmetric				
Reflection				

Solution:

Type	$\det(A) = 1$	A^2 is same Type	always real eigenvalues	diagonalizable
Rotation	*	*		*
Projection		*	*	*
Symmetric		*	*	*
Reflection			*	*

b) (3 points) Assume v is an eigenvector to A with eigenvalue 3 and A is invertible. Then

Statement	True	False
A^5 has an eigenvalue 3^5		
A^{-1} has an eigenvalue $1/3$		
A^T has an eigenvalue $1/3$		
A^5 has an eigenvector v		
A^{-1} has an eigenvector v		
A^T has an eigenvector v		

Solution:

Statement	True	False
A^5 has an eigenvalue 3^5	*	
A^{-1} has an eigenvalue $1/3$	*	
A^T has an eigenvalue $1/3$		*
A^5 has an eigenvector v	*	
A^{-1} has an eigenvector v	*	
A^T has an eigenvector v		*

c) (3 points) Which of the following statements are true about a real eigenvalue λ of a real matrix A?

Statement	True	False
If $A = A^T$ then the geometric and algebraic multiplicities of λ agree		
If $A \neq 1$ is a shear, then the geometric and algebraic multiplicity of λ do not agree		
The geometric multiplicity of λ is always positive		
The algebraic multiplicity of λ is always positive		
The algebraic multiplicity of λ can be smaller than the geometric multiplicity		
If $x(t+1) = Ax(t)$ is stable and B is similar to A then $x(t+1) = Bx(t)$ is stable		

Solution:
Only the second last is False

d) (2 points) Which statements are true for real matrices A, B?

Statement	True	False
If A is similar to B and A is symmetric then B is symmetric		
A diagonal matrix A is always symmetric		
A symmetric matrix A is always diagonalizable		
A symmetric matrix A always has real eigenvalues		

Solution:
Only the first statement is False.

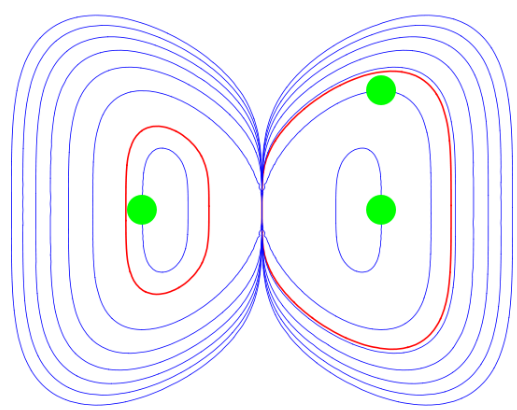
Problem 4) (10 points)

We want to model some data points with a quartic curve of the form

$$x^4 + y^4 + ax^2 - bx = 0 ,$$

where a, b are unknown parameters. Find the best linear fit to the following data points:

$$\begin{aligned}(x, y) &= (1, 1) \\ (x, y) &= (-1, 0) \\ (x, y) &= (1, 0) .\end{aligned}$$



Solution:

We have to find the least square solution to $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -2 \\ -1 \\ \\ -1 \end{bmatrix}$. We have $A^TA = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$. Plugging into the solution $(A^TA)^{-1}A^Tb = \begin{bmatrix} -5/4 \\ 1/4 \end{bmatrix}$.

Problem 5) (10 points)

The sequence $1, 1, 4, 7, 16, 31, ..$ is obtained from the recursion $x(t+1) = x(t) + 2x(t-1) + 1$. This can be written as a discrete dynamical system $v(t+1) = Av(t)$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, v(t) = \begin{bmatrix} x(t) \\ x(t-1) \\ 1 \end{bmatrix}.$$

a) (3 points) Knowing that $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$ are eigenvectors, find the eigenvalues of A.

b) (7 points) The initial condition $\vec{v}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ gives us the above sequence. Write down the explicit expression for $v(t) = A^tv(0)$.

Solution:

The sequence is called A051049 (https://oeis.org/).

a) Two eigenvalues of the matrix can be deduced from the eigenvectors, they are $2, -1, 1$.

b) We have to write the initial condition $\vec{v}(0)$ as a linear combination of the eigenvectors. We have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$. Therefore, $\vec{v}(t) = 2^t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + (-1)^t \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} + t^t \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

Problem 6) (10 points)

Decide in the following cases whether the dynamical system $v(t+1) = Av(t)$ is asymptotically stable.

a) (2 points) $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 1 & 2 \\ 4 & 1 & 0 \end{bmatrix}$.

b) (2 points) $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix}$.

c) (2 points) $A = \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \end{bmatrix}$.

d) (2 points) Why is the following rule true for a 3×3 matrix: if A is stable then $|\det(A)| < 1$.

e) (2 points) Why is the following rule true for a 3×3 matrix: if A is stable then $|\text{tr}(A)| < 3$.

Solution:
a) The trace is 3. There is one eigenvalue which has absolute value at least 3. This is unstable.
b) The trace is 5. Again, because of e) this is unstable.
c) The eigenvalues are $0, 0$ and $3/4$.
d) If $\det(A) = \lambda_1\lambda_2\lambda_3 > 1$, then one of the eigenvalues has to be larger than 1 in absolute value. Turned around, if all $|\lambda_i| < 1$, then $|\det(A)| < 1$.
e) If the eigenvalues are smaller than 1 in absolute value, then the sum of the eigenvalues is smaller than 1 in absolute value. In short, $|\lambda_k| < 1$ implies $\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 \leq |\lambda_1 + |\lambda_2| + |\lambda_3| < 1 + 1 + 1 = 3$.

Problem 7) (10 points)

a) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 9 & 2 & 2 & 2 & 2 & 2 \\ 1 & 8 & 1 & 1 & 1 & 1 \\ 1 & 1 & 8 & 1 & 1 & 1 \\ 1 & 1 & 1 & 8 & 1 & 1 \\ 1 & 1 & 1 & 1 & 8 & 1 \\ 2 & 2 & 2 & 2 & 2 & 9 \end{bmatrix}.$$

b) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

c) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

d) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

Solution:

a) Subtract 7 to see 5 eigenvalues 0 and one eigenvalue 8. Now add 7 to get 5 eigenvalues 7 and one eigenvalue 15. The answer is $7^5 \cdot 15$.
b) Partition. $4!^2 = 576$.
c) One pattern 1 with $6+5+4+3+2+1 = 21$ upcrossings. The determinant is -1.
d) This is a tough determinant.

If $Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, then $A = 1+5Q+4Q^2+3Q^3+2Q^4+Q^5$. Since the eigenvalues are known we get the product of $1+5\lambda+4\lambda^2+3\lambda^3+2\lambda^4$ where $\lambda = e^{2\pi ik/5}$. The answer can be simplified to 1875.

An other possibility is row reduce: subtract previous rows then subtract $5I_5$ to get $B = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$ which has 3 eigenvalues 0 and two eigenvaluies $-2 \pm i\sqrt{6}$.

The eigenvalues of B are therefore $5, 5, 5, (3+i\sqrt{6}), (3-i\sqrt{6})$. The determinant is then $5^3(3+i\sqrt{6})(3-i\sqrt{6}) = 5^3 \cdot 15 = 1875$.

Here is a third solution: row reduce B further by again subtracting the previous row:

$C = \begin{bmatrix} -5 & 5 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 \\ 2 & 3 & 4 & 0 & 6 \\ 4 & -1 & -1 & -1 & -1 \end{bmatrix}$ which has the same determinant and for which we can compute the determinant by looking at patterns.

Problem 8) (10 points)

Find the possibly complex eigenvalues and eigenbasis for the following matrices:

a) (2 points)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

b) (2 points)

$$B = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

c) (2 points)

$$C = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

d) (2 points)

$$D = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

e) (2 points)

$$E = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} .$$

Solution:
a) 3 with eigenvector e_1 and 3 with eigenvector e_2. Note that any non-zero vector is an eigenvector as the matrix is just 3 times the identity matrix.
b) 5 with eigenvector $[-1, 2]^T$ and -5 with eigenvector $[2, 1]^T$. This is a reflection dilation.
c) $3 \pm i4$ with eigenvectors $[\pm i, 1]$. d) 7 with eigenvector $[1, 1]$ and -1 with eigenvector $[-1, 1]^T$.
e) $\lambda_k = 2e^{2\pi ik/4}$ with $k = 1, 2, 3, 4$ and $v_k = [\lambda_k^3, \lambda_k^2, \lambda_k, 1]^T$.

Problem 9) (10 points)

Find the QR decomposition of the same matrices as in the previous problem (Déjà-vu):

a) (2 points)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

b) (2 points)

$$B = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

c) (2 points)

$$C = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

d) (2 points)

$$D = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

e) (2 points)

$$E = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} .$$

Solution:

a)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

b)

$$\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$$

c)

$$\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

d)

$$\begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix} \begin{bmatrix} 5 & 24/5 \\ 0 & 7/5 \end{bmatrix}$$

This was the only matrix, where we could not just write down the answer.

e)

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$