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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) T F A reflection is its own inverse.

Solution:
Yes, because $A^2 = I_n$ we have $A = A^{-1}$.

2) T F Every basis of R^n has the same number of vectors in it.

Solution:
We have shown that. It is called the dimension.

3) T F The rank of a 3×7 matrix is smaller or equal than 3.

Solution:
We can not have more than 3 leading 1's because each leading one is in its own column.

4) T F If $\{v_1, v_2, v_3, v_4\}$ is a set of linearly independent vectors of a linear space V, then $\dim(V) \geq 4$.

Solution:
v_1, v_2, v_3, v_4 is a basis of V.

5) T F If A is a 4×5 matrix, then the dimension of $\ker(A)$ is at least one.

Solution:
By the rank-nullety the theorem

6) T F The sum $A + B$ of 2 invertible matrices A, B is invertible.

Solution:
We can take $A = I_n$ and $B = -I_n$. Then the sum is not invertible.

7) T F If A, B, and C are $n \times n$ matrices, then the property $A(B+C) = AB + AC$ holds.

Solution:
This is a basic property of matrix multiplication

8) T F If $A\vec{x} = \vec{0}$ has no nonzero solutions, where A is a 4×3 matrix, then $\text{rank}(A) = 3$.

Solution:
The image can be 3 dimensional in R^4 and the vector b away from the image.

9) T F If $\vec{b}$ is in $\text{im}(A)$, then $A\vec{x} = \vec{b}$ exactly one solution.

Solution:
There can be a kernel.

10) T F If A is a nonzero column vector with 2 components and B is the same vector written as a row vector then AB is invertible.

Solution:
the resulting product is a 2×2 matrix which is not invertible. Any vector perpendicular to A is in the kernel of AB.

11) T F If $\vec{v}$ is a redundant column vector of A, then $\vec{v}$ is in the kernel of A.

Solution:
tests relationship between redundant vectors and kernel; common misconception

12) T F If $AB = I_n$ for an $n \times m$ matrix A and B is a $m \times n$ matrix, then $BA = I_m$.

Solution:
This is already not true for $A = [1, 0], B = [1, 0]^T$ where $AB = I_1$ and BA is a projection.

13) [**T**] [F] The product of 2 invertible 3×3 matrices is invertible.

Solution:
$(AB)^{-1} = B^{-1}A^{-1}$.

14) [**T**] [F] Suppose A and B are $n \times n$ matrices. If A is invertible and B is not, then AB is not invertible.

Solution:
The matrix B has a kernel. If v is a vector in this kernel, then $ABv = 0$ too and AAB has a kernel.

15) [T] [**F**] If a 2×2 matrix different from the identity is its own inverse then it is a reflection at a line.

Solution:
It can be a reflection at a point.

16) [**T**] [F] If a linear transformation from R^n to R^n has no nontrivial kernel, then it is invertible.

Solution:
Yes, then the rank is n and the row reduction produces the identity matrix.

17) [T] [**F**] The set of quadratic polynomials $ax^2 + bx + c$ has a basis consisting of 2 vectors.

Solution:
It has a basis of 3 vectors.

18) [T] [**F**] The set of vectors (x, y) in R^2 such that $xy > 0$ is a linear subspace of R^2.

Solution:
It does not contain the 0 vector.

19) T F The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is perpendicular to the vector $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

Solution:
The dot product is zero

20) T F The solution set to the system of equations $x+y=1, 2x+2y=2$ is a linear space.

Solution:
The set does not contain the zero vector $(0,0)$.

	Total

Problem 2) (10 points) No justifications are needed.

a) (3 points) One of the following matrices can be composed with a dilation to become an orthogonal projection onto a line. Which one?

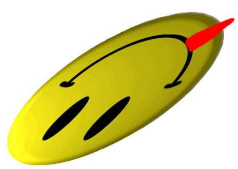
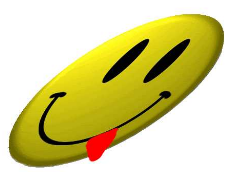
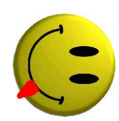
$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

b) (4 points)
The **smiley face** visible to the right is transformed with various linear transformations represented by matrices $A - F$. Find out which matrix does which transformation:

$$\text{A=} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{B=} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{C=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\text{D=} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad \text{E=} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{F=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}/2$$

A-F	image	A-F	image	A-F	image
					
					

c) (3 points) Which of the following sets are linear spaces?

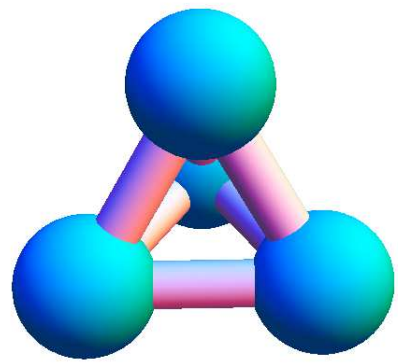
Space	Check
the image of the identity matrix	
the kernel of the identity matrix	
the unit circle in the plane	

Space	Check
the image of $(x, y, z) \to x^2$	
points (x, y) with $3x^2 + 2y = 0$	
the reals satisfying $x^2 = 0$	

Solution:
a) the matrix D is a projection onto a line, if we divide by 2.
b) upper row D C E, lower row: A,B,F
c) The circle and the parabola $3x^2 + 2y = 0$ are not linear spaces.

Problem 3) (10 points)

A **tetrahedral molecule** has four corners with charges x, y, z, w. We know the sum of the charges of three of the four faces and want to determine the individual charges. Find all possible charge distributions.



$$\left| \begin{array}{ccccccccc} x & + & y & + & z & & & = & 12 \\ x & + & y & & & + & w & = & 9 \\ x & + & & & z & + & w & = & 8 \end{array} \right|$$

Solution:
We row reduce the augmented matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 12 \\ 1 & 1 & 0 & 1 & 9 \\ 1 & 0 & 1 & 1 & 8 \end{bmatrix}$$

to get

$$\text{rref}(B) = \begin{bmatrix} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & 3 \end{bmatrix} .$$

There is 1 free variables, $w = t$. The general solution is $x = 5-2t, y = 4+t, z = 3+t, w = t$. We can also write it as a special solution plus a multiple of a kernel

$$\begin{bmatrix} 5 \\ 4 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix} .$$

Problem 4) (10 points)

Find a basis of the image and a basis for the kernel of the following **Pascal triangle matrix**:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \end{bmatrix} .$$

Solution:
We produce the row reduced echelon form of the matrix:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

The kernel has the basis

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} .$$

The image is spanned by the first 5 vectors of the original matrix, because these are the pivot columns, the columns with leading 1.

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 6 \end{bmatrix} \right\} .$$

Problem 5) (10 points)

a) (5 points) Invert the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

using row reduction.
b) (5 points) Find a basis for the space of vectors perpendicular to the image of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} .$$

Solution:
a)

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

b) We want to find the kernel of the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} ,$$

which has the column vectors of A as row vectors. The row reduced echelon form is

$$\text{rref}(B) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

There are 2 free variables. We call them s,t. The kernel is $x = s - t, y = s, z = 0, w = t$, so that

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the set of vectors we were looking for.

Problem 6) (10 points)

Let T be the linear transformation from R^3 to R^3 which is an orthogonal projection onto the plane

$$2x - y + 5z = 0 .$$

In the standard basis, we have $T(x) = Ax$.

a) (2 points) What is the dimension of $\ker(A)$?
b) (4 points) Find a basis $\mathcal{B}$ so that T is described by a diagonal matrix B.

c) (2 points) Find that diagonal matrix B.
d) (2 points) Which of the following relationships between A and B hold? Check all which apply:

Property	Check if true
A is similar to B	
$AB = BA$	
$SA = BS$ for some matrix S	
$A = MBM^{-1}$ for some matrix M	

Solution:
a) Since the image is two dimensional, by the rank-nullety theorem, the dimension of the kernel is equal to $1 = 3 - 2$.
b) Take one vector perpendicular to the plane: $\vec{v}_1 = \langle 2, -1, 5 \rangle$ and two vectors in the plane like $\vec{v}_2 = \langle 1, 2, 0 \rangle$ and $\vec{v}_3 = \langle 0, 5, 1 \rangle$.
c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
d)

Property	Check if true
A is similar to B	True, since $B = S^{-1}AS$.
$AB = BA$	False, in general
$SA = BS$ for some matrix S	True, since $B = S^{-1}AS$.
$A = MBM^{-1}$ for some matrix M	True, $M = S^{-1}$

We did not intend this to be possible but also $S = 0$ would give a reason for the third question.

Problem 7) (10 points)

Let $S(\vec{x}) = A\vec{x}$ be the transformation given by

$$A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} /2$$

and $T(\vec{x}) = B\vec{x}$ be the dilation (=scaling) transformation by the dilation factor $\frac{1}{2}$. Finally, let $R = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ be a horizontal shear.

a) (4 points) If $M(\vec{x}) = T(S(\vec{x}))$, find the matrix which gives the transformation

$$M^{13} = M(M(M(M(M(M(M(M(M(M(M(M(M(\vec{x}))))))))))))) \, .$$

b) (4 points) Let $U(\vec{x}) = R\vec{x}$ where If $L(\vec{x}) = U(S(\vec{x}))$, find the matrix which gives the transformation L.
c) (2 points) Is $AR = RA$?

Solution:

a) T is a rotation-dilation. M^{13} which is a composition of a rotation by 13 times 30 degrees and 13 times a scaling 1/2. The result is a rotation by 30 degrees composed by a scaling by $1/2^{13}$.

$M^{13} = \left[\begin{array}{cc} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{array} /2^{14} \right]$.

b) $L = RA = \left[\begin{array}{cc} \sqrt{3}+2 & -1+2\sqrt{3} \\ 1 & \sqrt{3} \end{array} \right] /2$.

c) No, the two do not commute: $AR = \left[\begin{array}{cc} \sqrt{3} & 2\sqrt{3}-1 \\ 1 & 2+\sqrt{3} \end{array} \right] /2$.

Problem 8) (10 points)

a) (5 points) What are the coordinates of the vector $\left[\begin{array}{c} 4 \\ 3 \\ 3 \end{array} \right]$ in the basis

$$\mathcal{B} = \{ \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right], \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] \} \ ?$$

b) (5 points) A transformation T is described in the standard basis by

$$A = \left[\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right] .$$

What is the matrix B of the transformation T in the basis $\mathcal{B}$?

Solution:
a) The matrix S which contains the basis vectors as columns is

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} .$$

Its inverse is

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} /2 .$$

The vector $\vec{v}$ in the $\mathcal{B}$ coordinates is $[\vec{v}]_{\mathcal{B}} = S^{-1}v = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

b) The matrix B in the basis $\mathcal{B}$ is

$$B = S^{-1}AS = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} /2 = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix} /2 .$$

Problem 9) (10 points)

In this problem, we consider the following four matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}, C = \begin{bmatrix} -2 & 2 \\ 2 & 2 \end{bmatrix}, D = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} .$$

a) (5 points) For each of the matrices A, B, C, find the row reduced echelon form and the rank. Decide further whether the matrix is invertible and whether the matrix is similar to the matrix D:

matrix M	rref(M)=	rank(M) =	invertible?	similar to D?
A	$\begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$			
B	$\begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$			
C	$\begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$			

b) (5 points) One of the matrices A, B, C is a reflection-dilation (a reflection composed with a dilation), one is a projection dilation (a projection composed with a dilation) and one is a rotation dilation (a rotation composed with a dilation). Identify which is which. Find the dilation factor. For the reflection dilation matrix, determine the line about which the reflection takes place. For the projection dilation matrix, find the line onto which it projects. For the rotation dilation matrix determine the angle of rotation.

Type	Matrix (enter A,B,C)	dilation factor	angle or line
rotation dilation			angle=
projection dilation			projection line =
reflection dilation			reflection line =

Solution:

a)

$$\text{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\text{rref}(B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\text{rref}(C) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

All except the first are invertible. The rank of A is 1, the rank of the others is 2. Only C is similar to D. The matrix B is a rotation-dilation matrix.

matrix M	rref(M)=	rank(M) =	invertible?	similar to D?
A	$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$	1	no	no
B	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	2	yes	no
C	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	2	yes	yes

b)

Type	Matrix (enter A,B,C)	dilation factor	angle or line
rotation dilation	B	$2\sqrt{2}$	angle= 45 degrees $= \pi/4$
projection dilation	A	2	projection line $= x = y$
reflection dilation	C	$2\sqrt{2}$	reflection line = line with angle $3\pi/8$

How can one get the dilation factor? For the rotation dilation $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, the factor is $\sqrt{a^2+b^2}$. Similarly, for a reflection dilation $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. For a projection, we have to place a vector into the image and see how much longer it gets. In the case of the projection A, we can take a vector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The vector Av is twice as long so that the dilation factor is 2.

Problem 10) (10 points)

Define

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & -5 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

a) (2 points) Find the row reduced echelon form of A and the rank.

b) (2 points) What is the rank of A and what is the nullity of A?

c) (2 point) Find b such that $Ax = b$ has infinitely many solutions or state if there is none.

d) (2 point) Find b such that $Ax = b$ has one solution or state if there is no such vector.

e) (2 point) Find b such that $Ax = b$ has no solution or state why there is no such vector.

Solution:
a)

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) The rank is 3, the nullity is 2.
c) Take b as the first column of A.
d) There is no such case.
e) Start with the augmented row reduced matrix and put in the last column a vector so that we have no solution, then row reduce back

$$\begin{bmatrix} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The vector e_1 does the job.