Checklist second hourly — Math 21b, Spring 2015 O. Knill

Main Skills

- [ ] **Computing Determinants** (rref,Laplace,patterns,eigenvalues,partition).
- [ ] **Gram-Schmidt Orthogonalization** (QR decomposition).
- [ ] **Computing Eigenvalues of A** (factor $f_A(\lambda)$, trace, noninvertibility).
- [ ] **Computing Eigenvectors of** A (determining kernel of $A - \lambda I_n$).
- [ ] **Computing algebraic and geometric multiplicities**
- [ ] **Find the orthogonal projection onto a subspace**.
- [ ] **Produce the least square solution of a linear equation.**
- [ ] **Solve data fitting problem** (linear system, least square problem).
- [ ] **Algebra of matrices** (multiplication, inverse, transpose).
- [ ] **Diagonalize Matrix** (find the eigensystem to produce S).
- [ ] **Decide about similarity** (for example by diagonalization).
- [ ] **Solve discrete systems** (eigensystem, write v in eigenbasis).
- [ ] **Calculate with complex numbers** (add, multiply, divide, take roots)
- [ ] **Find orthonormal eigenbasis** (possible if $A = A^T$ by spectral theorem)

Definitions

- [ ] **Orthogonal vectors** $\vec{v} \cdot \vec{w} = 0$.
- [ ] **Length** $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$, **Unit vector** $\vec{v}$ with $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = 1$.
- [ ] **Orthogonal set** $v_1, \dots, v_n$: pairwise orthogonal.
- [ ] **Orthonormal set** orthogonal and length 1.
- [ ] **Orthonormal basis** A basis which is orthonormal.
- [ ] **Orthogonal to V** v is orthogonal to V if $v \cdot x = 0$ for all $x \in V$.
- [ ] **Orthogonal complement in $\mathbf{R}^n$** space $V^\perp = \{v \in \mathbf{R}^n \mid v \text{ perpendicular to } V\}$.
- [ ] **Projection onto V** orth. basis $P = QQ^T$ if Q has orthonormal columns.
- [ ] **Gram-Schmidt** $u_i = v_i - \text{proj}_{V_{i-1}} v_i$, $w_i = u_i/||u_i||$
- [ ] **QR-factorization** $Q = [w_1 \cdots w_n]$, $R_{ii} = |u_i|$, $R_{ij} = w_i \cdot v_j, j > i$.
- [ ] **Transpose** $A^T_{ij} = A_{ji}$. Transposition switches rows and columns.
- [ ] **Symmetric** $A^T = A$ and **skew-symmetric** $A^T = -A$

Orthogonal matrix $A^T A = 1$ (i.e. rotations or reflections)

Orthogonal projection onto V is $A(A^T A)^{-1} A^T$.

Normal equation to $Ax = b$ is the consistent system $A^T Ax = A^T b$.

Least square solution of $A\vec{x} = \vec{b}$ is $\vec{x}_* = (A^T A)^{-1} A^T \vec{b}$.

Angle between two vectors x, y is $\alpha = \arccos((x \cdot y)/(||x|| \, ||y||))$.

Determinant $\det(A) = (\sum_{even\ \pi} - \sum_{odd\ \pi}) A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)}$.

Orientation $\text{sign}(\det(A))$ defines orientation of basis given by columns of A.

Trace is $\text{tr}(A) = \sum_i A_{ii}$, the sum of diagonal elements of A.

Characteristic polynomial $f_A(\lambda) = \det(A - \lambda I_n)$.

Char. Polynomial $f_A(\lambda) = (-\lambda)^n + \text{tr}(A)(-\lambda)^{n-1} + \ldots + \det(A)$.

Eigenvalues and eigenvectors $Av = \lambda v, v \neq 0$.

Factorization. Have $f_A(\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$ **with roots** λ_i**.**

Algebraic multiplicity is k, if $f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$ with $g(\lambda_0) \neq 0$.

Geometric multiplicity the dimension of the kernel of $A - \lambda I_n$.

Kernel and eigenspace $\ker(A - \lambda)$ is eigenspace.

Eigenbasis Basis which consists of eigenvectors of A.

Complex numbers $z = x + iy = |z| \exp(i\,\theta) = r\cos(\theta) + ir\sin(\theta)$.

Modulus $|z| = |x + iy| = \sqrt{x^2 + y^2}$

Argument $\theta = \arg(z) = \arctan(y/x)$, polar angle in the complex plane.

Conjugate complex number $\overline{z} = x - iy$ if $z = x + iy$.

Root $1^{1/n} = e^{2\pi ik/n}$, $k = 1, \ldots, n$ gives complex numbers on regular n-gon.

Linear dynamical system vector equation $\vec{v}(t + 1) = A\vec{v}(t)$ with initial condition $v(0)$ known.

Asymptotic stability $A^n \vec{x} \to 0$ for all $\vec{x}$ (equivalent $|\lambda_i| < 1$)

Some facts

Orthogonal complement With notation $V^\perp$, have $(V^\perp)^\perp = V$.

Projection property $\vec{x} - \text{proj}_V(\vec{x})$ is orthogonal to V.

Pythagoras x, y orthogonal $\Rightarrow ||x + y||^2 = ||x||^2 + ||y||^2$.

Complement of Image of A $\ker(A^T) = (\text{im}(A))^\perp$.

Dimension of orthogonal complement $\dim(V) + \dim(V^T) = n$.

Cauchy-Schwartz inequality $|x \cdot y| \leq ||x|| \, ||y||$.

Triangle inequality $||x+y|| \leq ||x|| + ||y||$.

Row vectors of A are orthogonal to $\ker(A)$. Short: $\mathrm{im}(A^T) = \ker(A)^\perp$.

Orthogonal transformations preserve angle, length, columns are orthonormal

Orthogonal projection $P = A(A^TA)^{-1}A^T$ onto $\mathrm{im}(A)$.

Orthogonal projections Only identity is both orthogonal projection and orthogonal transformation

Kernel of A and A^TA are the same: $\ker(A) = \ker(A^TA)$.

Determinant $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$.

Det $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - fha - bdi$. Sarrus rule.

Pattern: Determinant is sum of patterns. Odd up-crossings counted negatively.

Det of triangular matrix product of diagonal entries.

Det of partitioned matrix $\det(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}) = \det(A)\det(B)$.

Product properties $\det(AB) = \det(A)\det(B)$, $\det(A^{-1}) = 1/\det(A)$.

Determinants $\det(SAS^{-1}) = \det(A)$, $\det(A^T) = \det(A)$.

Adding rows or columns $\det([A\vec{a}B]) + \det([A\vec{b}B]) = \det([A(\vec{a}+\vec{b})B])$

Scaling rows or columns $\det([A(\lambda\vec{a})B]) = \lambda\det([A\vec{a}B])$.

Swapping of two rows $\det(B) = -\det(A)$.

Adding row to given row $\det(B) = \det(A)$

Rref $\det(A) = (-1)^s(\prod_i c_i^{-1})\det(\mathrm{rref}(A))$, c_i scalings and s swappings.

Properties of transpose $(A^T)^T = A$, $(AB)^T = B^TA^T$, $A^T + B^T = (A+B)^T$

Properties of transpose $(A^{-1})^T = (A^T)^{-1}$, $\det(A^T) = \det(A)$.

Diagonalization S has eigenvectors of A in columns, $S^{-1}AS$ diagonal.

Nontrivial kernel $\Leftrightarrow \det(A) = 0 \Leftrightarrow$ have eigenvalue 0.

Invertible Matrix $\Leftrightarrow \det(A) \neq 0$.

Laplace Expansion $\det(A) = (-1)^{i+1}a_{i1}\det(A_{i1}) + \cdots + (-1)^{i+n}a_{in}\det(A_{in})$

Orthogonal Matrices A have $\det(A) = \pm 1$

Rotations satisfy $\det(A) = 1$ in all dimensions.

Rotations with angle α in the plane have eigenvalues $\exp(i\alpha) = \cos(\alpha) + i\sin(\alpha)$.

QR Decomposition $A = QR$ orthogonal A, upper triangular R.

QR Decomposition $|\det(A)| = \prod_{i=1}^{n} R_{ii}$ if $Q = QR$.

Determinant is Product of eigenvalues. $\det(A) = \prod_i \lambda_i$.

Trace is Sum of eigenvalues. $\text{tr}(A) = \sum_i \lambda_i$.

Geometric Multiplicity $\leq$ Algebraic Multiplicity.

Simple Eigenvalues $\Rightarrow$ Diagonalize, if all algebraic multiplicities 1

Eigenvalues of A^T agree with eigenvalues of A (same $f_A(\lambda)$).

Rank of A^T is equal to the rank of A.

Reflection at k-dimensional subspace of $\mathbf{R^n}$ has det $(-1)^{(n-k)}$.

Euler $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

De Moivre $z^n = \exp(in\theta) = \cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n$.

Fundamental Theorem of Algebra $f_A(\lambda)$ has exactly n roots.

Number of eigenvalues A $n \times n$ matrix has exactly n eigenvalues.

Power of a Matrix A^k has eigenvalues λ^k if A has eigenvalue λ.

Eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\lambda_\pm = \text{tr}(A)/2 \pm \sqrt{(\text{tr}(A)/2)^2 - \det(A)}$.

Eigenvectors of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $c \neq 0$ are $v_\pm = \begin{bmatrix} \lambda_\pm - d, c \end{bmatrix}^T$.

Rotation-Dilation Matrix $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$, eigenvalues $p \pm iq$

Rotation-Dilation Matrix eigenvectors $[\pm i, 1]^T$.

Permutation Matrix $Ae_i = e_{i+1}$, roots on regular polygon.

Data Fitting Least square solution of $A\vec{x} = \vec{b}$, where A and b are given by data.

Similarity A, B with simple eigenvalues are similar if eigenvalues are the same

Similarity If A, B are similar, then A^n, B^n are similar

Similarity If A, B are similar, then eigenvalues, trace and det are the same

Similarity If some λ_j of A^k has different geometric multiplicity, then not similar.