

Homework 26: Nonlinear systems

This homework is due on Wednesday, April 15, respectively on Thursday, April 16, 2015. These problems are adapted from the problems in the handout.

1 Analyze the system

$$\begin{aligned}\frac{dx}{dt} &= 2x - x^2 + xy \\ \frac{dy}{dt} &= 4y - xy - y^2\end{aligned}$$

It is an interaction model of species so that we only look at $x \geq 0, y \geq 0$.

2 Analyze the system

$$\begin{aligned}\frac{dx}{dt} &= x(1 - x + ky - k) \\ \frac{dy}{dt} &= y(1 - y + kx - k)\end{aligned}$$

where k is a constant different from 1, -1 . Again, since this is a population model, we only look at $x \geq 0, y \geq 0$.

3 Analyze the frictionless pendulum

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -c \sin(x),\end{aligned}$$

where c is a positive constant.

4 Analyze the system

$$\begin{aligned}\frac{dx}{dt} &= x^2 + y^2 - 1 \\ \frac{dy}{dt} &= xy\end{aligned}$$

5 Analyze the pendulum with friction

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\sin(x) - y.\end{aligned}$$

Nonlinear systems

We look at differential equations $x' = f(x, y)$, $y' = g(x, y)$. This generalizes the linear case $x' = ax + by$, $y' = cx + dy$. To analyze such systems, we draw phase portraits. The curves where $f(x, y) = 0$ or $g(x, y) = 0$ are called nullclines. They intersect in **equilibrium points**. These are points where $x' = 0$, $y' = 0$. We can use linear algebra to analyze the system near such an equilibrium point (a, b) . The matrix $A = \begin{bmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{bmatrix}$ is called the **Jacobian matrix**. The linear system $v' = Av$ is called the **linearization** at (x_0, y_0) . If this linear system is stable, the equilibrium point is stable. In terms of the original nonlinear system, an equilibrium point (x_0, y_0) is stable if all trajectories starting sufficiently close to (x_0, y_0) tend to it as $t \rightarrow \infty$.

Making an **analysis** of the system consists of 1) finding the nullclines and equilibria 2) determine the stability of the equilibria 3) drawing the phase portrait of the system 4) analyzing the possible behaviors of the trajectories.