

Homework 25: Differential equations II

This homework is due on Monday, April 13, respectively on Tuesday, April 14, 2015.

1 Find the solution to the differential equation $\frac{d^2x}{dt^2} = -4x$ with initial conditions $x(0) = 5, \frac{dx}{dt}(0) = 3$.

2 a) Verify that the matrix $e^{At} = 1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$ has the property that $\frac{d}{dt}e^{At} = Ae^{At}$. You can assume that the derivative of the infinite sum is the sum of the derivatives of the individual terms.

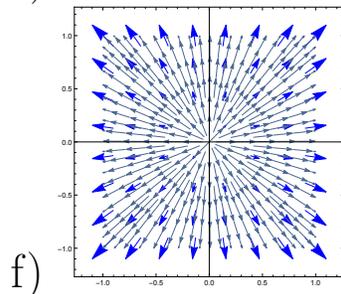
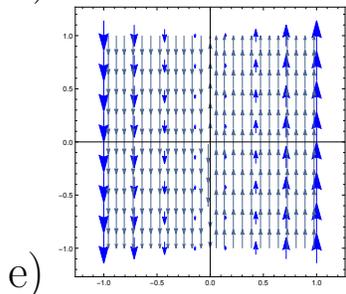
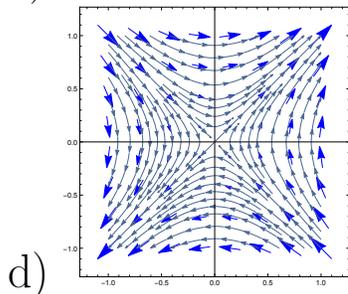
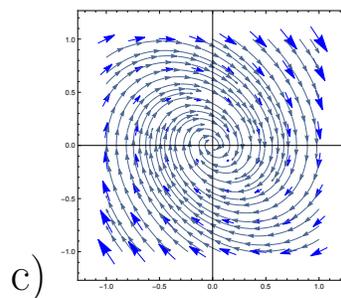
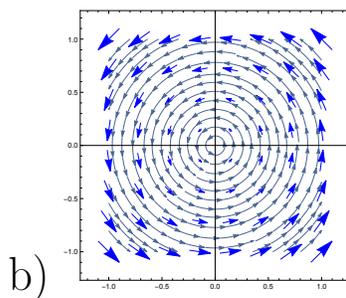
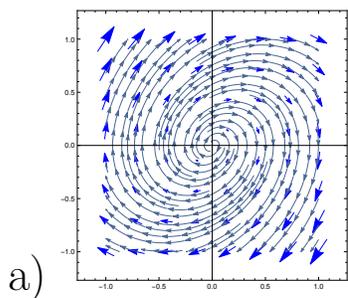
b) Verify that $x(t) = e^{At}x(0)$ is a solution to the differential equation $\frac{d}{dt}x(t) = Ax(t)$.

c) Compute e^{At} if $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. To do so, find first an expression for A^n , then use the series formula in a).

3 Match the differential equations $\frac{dx}{dt} = Ax$ with the phase portraits.

i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, ii) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ iii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

iv) $A = \begin{bmatrix} 0 & 1 \\ -1 & 1/2 \end{bmatrix}$ v) $A = \begin{bmatrix} 0 & 1 \\ -1 & -1/2 \end{bmatrix}$ vi) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$



4 Determine the stability of the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix} x(t).$$

5 True or false? Give short explanations

- a) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = A^T x$ is stable.
- b) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = A^{-1}x$ is stable.
- c) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = -Ax$ is stable.
- d) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = (A + I_n)x$ is stable.
- e) If $\frac{dx}{dt} = Ax$ is stable, then $\frac{dx}{dt} = (A - I_n)x$ is stable.

Differential Equations II

The equations $\frac{d^2x}{dt^2} = -k^2x(t)$ is called the **harmonic oscillator**. the solution is of the form $x(t) = a \cos(kt) + b \sin(kt)$, where a, b are determined by the initial conditions. We can turn it into a system by rewriting it as $\frac{dx}{dt} = y(t), \frac{dy}{dt} = -k^2x(t)$. For a system $\frac{dx}{dt} = Ax$ with 2×2 matrix A , the system is stable if and only if the determinant is positive and the trace is negative.

If A is a matrix which is diagonalizable there is an easier way to compute e^{At} than by the power series in problem 2a): if $A = SBS^{-1}$, with B diagonal, then $e^{At} = Se^{Bt}S^{-1}$; and e^{Bt} is the diagonal matrix with diagonal entries $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ where $\lambda_1, \dots, \lambda_n$, are the diagonal entries of B (the eigenvalues of A).