Homework 24: Differential equations I

This homework is due on Friday, April 9, respectively on Tuesday, April 14, 2015.

1 a) Solve the differential equation $\frac{dx}{dt} = 1/x$,with $x(0) = 1$.
b) Solve the differential equation $\frac{dx}{dt} = 1 + x^2$,with $x(0) = 0$.
c) Solve the differential equation $\frac{dx}{dt} = 1/\cos(x)$,with $x(0) = 0$.

2 Solve the system

$$\frac{dx}{dt} = Ax, \quad A = \begin{bmatrix} 3 & 5 \\ 4 & 4 \end{bmatrix}$$

with initial condition $x(0) = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$.

3 For which p, q is the system $\frac{dx}{dt} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} x(t)$ stable?

4 The interaction of two animal species is modeled by the equations

$$\frac{dx}{dt} = 1.5x - 1.2y$$
$$\frac{dy}{dt} = 0.8x - 1.4y$$

a) Interpret the system. Is it a symbiosis, competition or predator-prey?
b) Sketch the phase portrait in the first quadrant.
c) What happens in the long term? Does it depend on the initial population? If so, how?

5 A door opens on one side only. A spring mechanism closes the door which forms an angle $\theta(t)$ with the frame. The angular velocity

is $\omega(t) = \frac{d\theta}{dt}(t)$. The differential equations are

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -2\theta - 3\omega$$

The first equation is the definition, the second incorporates the force -2θ of the spring and the friction -3ω.

Sketch a phase portrait for the system and use this to answer the question, for which initial conditions, the door slams (reaches $\theta = 0$ with negative ω).

Differential Equations I

A system $\frac{dx}{dt} = f(x)$ is a differential equation. One can often solve them by separation of variables. For example, if $\frac{d}{dt}x = x^2/t$, $x(0) = 0$, then we get $tdt = x^2dx$ and integrate both sides to get $t^2/2 = x^3/3 + c$ so that $x(t) = (3(t^2/2 - c))^{1/3}$. As $x(0) = 0$ we have $c = 0$ and $x(t) = (3t^2/2)^{1/3}$. The linear differential equation $\frac{dx}{dt} = kx$ has the solution $x(t) = e^{kt}x(0)$. For $k > 0$, this gives exponential growth. For $k < 0$, exponential decay. A linear system of differential equations is $\frac{dx}{dt} = Ax$. If $x(0) = v$ is an eigenvector with eigenvalue λ, then $x(t)$ is always a multiple of v, say $x(t) = c(t)v$ where $\frac{dc}{dt} = \lambda v$. Thus if $x(0) = c_1v_1 + \ldots + c_nv_n$ writes an initial condition as a sum of eigenvectors, then $x(t) = c_1e^{\lambda_1 t}v_1 + \ldots + c_ne^{\lambda_n t}v_n$ is the explicit solution of the system. A system is asymptotically stable, if $x(t) \to 0$ for all initial conditions $x(0)$. We have asymptotic stability if $\text{Re}(\lambda_j) < 0$ for all j. Be sure to compare all this with the case of discrete dynamical systems.