

Homework 21: Complex Eigenvalues

This homework is due on Wednesday, April 1, respectively on Thursday, April 2, 2015.

- 1 a) For $z = 3 + 4i$. Find $5z + z^3$.
- b) The log of a nonzero complex number $re^{i\theta} = r \cos(\theta) + ir \sin(\theta)$, where $r > 0$ and $0 \leq \theta < 2\pi$, is given by $\log(re^{i\theta}) = \log r + i\theta$. Find the logarithm of $e i$. ("Ei" is the product of e and i . It means "Egg" in German. Its soon easter!)
- c) Using logarithms we can define $w^z = e^{z \log w}$. What is i^i , the "eye for an eye" number?

- 2 Express $\cos(4\theta)$ and $\sin(4\theta)$ as polynomials in $\cos(\theta)$, $\sin(\theta)$.

- 3 a) Find all the complex eigenvalues of the matrix $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.
- b) Verify that if λ is an eigenvalue, $\vec{v} = \begin{bmatrix} \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$ is an eigenvector.

- 4 Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

Hint: Do you recognize $A - 3I_6$?

- 5 The matrix $A = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.8 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.0 \end{bmatrix}$ is called a Markov matrix: in every column the entries add up to 1, so each column can be interpreted as a probability distribution.

- a) Verify that A^T has the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with eigenvalue 1.
- b) Why does A also have an eigenvalue 1? Find all other eigenvalues.
- c) Find all eigenvectors of A .

Complex eigenvalues

Numbers of the form $z = a+ib$ are called complex numbers. One can add and multiply these numbers as real numbers, just keeping in mind that $i^2 = -1$. Euler found $e^{i\theta} = \cos(\theta) + i \sin(\theta)$ which in the special case $\theta = \pi$ gives $e^{i\pi} + 1 = 0$, which is by far voted the most beautiful formula in whole of mathematics as it combines $e, \pi, i, 1$ and 0 . As e is part of analysis and π is part of geometry and 0 is the additive neutral element and 1 the multiplicative neutral element, this identity combines analysis, geometry and algebra. The Euler identity leads to **de Moivre formulas** like $(\cos(\theta) + i \sin(\theta))^3 = (e^{i\theta})^3 = e^{i3\theta} = \cos(3\theta) + i \sin(3\theta)$. So that $\cos^3(\theta) - \cos(\theta) \sin^2(\theta) = \cos(3\theta)$ and $\cos^2(\theta) \sin(\theta) - \sin^3(\theta) = \sin(3\theta)$. Eigenvalues of matrices can become complex as the rotation-dilation matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

shows which has the eigenvalues $a \pm ib$. The fundamental theorem of algebra assures that the sum of the algebraic multiplicities of all eigenvalues of a $n \times n$ matrix is equal to n .