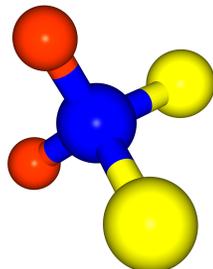# Homework 20: Diagonalization

This homework is due on Monday, March 30, respectively on Tuesday, March 31, 2015.

1 The **Freon molecule** (Dichlorodifluoromethane or shortly CFC-12) CCl_2F_2 has 5 atoms. It is a CFC was used in refrigerators, solvents and propellants but contributes to ozone depletion in the atmosphere. The adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) Verify that the matrix has the characteristic polynomial $-(x^5 - 4x^3)$.

b) Find the eigenvalues and eigenvectors of A. c) Write down a diagonal matrix B which is similar to A. What is the matrix S which does the diagonalization $B = S^{-1}AS$?

2 Which of the following matrices are diagonalizable? To find out, see whether there is an eigenbasis: a) $A = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$, b) $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, c) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, d) $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

3 Are the following matrices similar? $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hint: Compute A^2 and B^2 and use that if A and B are similar then A^2 and B^2 are similar.

4 Find $2^A + A^4 + A$ for $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}^{100}$ by diagonalization.

5 Let A be a 3×3 matrix for which $A^2 = 0$. Verify that the image is a subspace of the kernel of A. Find the dimension of the image and kernel. Pick v_1 in the image of A and write $v_1 = Av_2$. Let v_3 be a vector in the kernel which is not a multiple of v_1. a) Verify that $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis. Find the matrix B in that basis.

b) The matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix}$ is not diagonalizable. Using a) to

find S such that $S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Diagonalization

If A is similar to a diagonal matrix B, then A is called **diagonalizable**. In that case the coordinate transformation S has the eigenvectors of A as columns. A key result is that every $n \times n$ matrix which has n different eigenvalues is diagonalizable. The reason is that the eigenvectors form then an eigenbasis. If A is diagonalizable with diagonal matrix $B = S^{-1}AS$ one can define $f(A)$ for any function f by $f(A) = Sf(B)S^{-1}$ where f is applied to each diagonal entry of B. For example $\sin(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix})$ is $S \begin{bmatrix} \sin(0) & 0 \\ 0 & \sin(2) \end{bmatrix} S^{-1}$ with $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ which is $\begin{bmatrix} \frac{\sin(2)}{2} & -\frac{\sin(2)}{2} \\ -\frac{\sin(2)}{2} & \frac{\sin(2)}{2} \end{bmatrix}$.