

Homework 18: Eigenvalues

This homework is due on Wednesday, March 25, respectively on Thursday, March 26, 2015.

1 Show that $\begin{bmatrix} 14 & -10 \\ 20 & -16 \end{bmatrix}$ has an eigenvalue 4 and an eigenvalue -6 . Find the corresponding eigenvectors.

2 The matrix $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ is a reflection dilation. Use geometric insight to find the eigenvalues and eigenvectors.

3 A Lilac bush has $n(t)$ new branches and $o(t)$ old branches at the beginning of each year t . During the year, each old branch will grow two new branches and every new branch will become a old branch. a) Find the matrix A such that

$$\begin{bmatrix} n(t+1) \\ o(t+1) \end{bmatrix} = A \begin{bmatrix} n(t) \\ o(t) \end{bmatrix}$$

b) Verify that $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are eigenvectors. Find the eigenvalues.

c) Find closed formulas for $n(t)$, $o(t)$ if the initial condition $\begin{bmatrix} n(0) \\ c(0) \end{bmatrix} = c_1 v_1 + c_2 v_2$ are given. If you prefer to work with an example, take the initial condition $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

4 a) Find the characteristic polynomial of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and determine its roots. b) What are the eigenvalues and eigenvectors of the projection $P(x, y, z) = (x, y, 0)$ from space to the xy -plane?

- 5 a) Find all the eigenvalues with multiplicity of I_n .
 b) Verify that every $(2n+1) \times (2n+1)$ matrix has a real eigenvalue.
 c) Find a $2n \times 2n$ matrix for which there is no real eigenvalue.

Eigenvalues

A nonzero vector v is an **eigenvector** of A , if $Av = \lambda v$ for some real number λ called **eigenvalue**. A basis \mathcal{B} consisting of eigenvectors of A is called an **eigenbasis**. Eigenvalues λ_j and vectors v_j help to solve **discrete dynamical systems** $x \rightarrow Ax$, where we want to find closed formulas for the trajectories $A^t x$: write an initial vector x as a sum of eigenvectors $x = c_1 v_1 + \dots + c_n v_n$, then get $A^t x = c_1 \lambda_1^t v_1 + \dots + c_n \lambda_n^t v_n$. One can find the eigenvalues of a matrix A by finding the roots of the **characteristic polynomial** $f_A(\lambda) = \det(A - \lambda I_n)$. It is a polynomial of degree n of the form

$$f_A(\lambda) = (-\lambda)^n + \text{tr}(A)(-\lambda)^{n-1} + \dots + \det(A) .$$

The algebraic multiplicity of an eigenvalue is the multiplicity of the root. The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ for example has the characteristic polynomial

$$f_A(\lambda) = \det \begin{pmatrix} 1 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 - \lambda \end{pmatrix}$$

which is $-\lambda^3 + 3\lambda^2 = \lambda^2(3 - \lambda)$ showing that $\lambda = 0$ is an eigenvalue of algebraic multiplicity 2 and $\lambda = 3$ is an eigenvalue of multiplicity 1.