Homework 16: Determinants

This homework is due on Friday, March 13, respectively on Tuesday, March 24, 2015.

1 Find the determinants of A, B, C:

$$
A = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 4 & 7 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \end{bmatrix},\
B = \begin{bmatrix} 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 4 & 1 & 1 \\ 2 & 4 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \end{bmatrix},\
C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & 7 & 3 & 0 \\ 0 & 3 & 9 & 6 & 4 & 0 \\ 1 & 8 & 6 & 8 & 0 & 9 \end{bmatrix}
$$

2 Without doing much computation, determine whether the following determinant is positive, zero or negative:

$$
\begin{bmatrix}
1 & 1000 & 2 & -1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1000 & -1 \\
1000 & 1 & 0 & 1 & 0 & 1 \\
2 & 0 & 1000 & 1 & 1 & 1 \\
2 & 1 & -1 & 1000 & 2 & 0 \\
6 & 1 & -1 & 0 & 4 & 1000
\end{bmatrix}.
$$

3 a) The Leibniz definition of determinants shows directly that the **partitioned matrix** satisfies $\det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det(A)\det(B)$. Assume now that C, D are $n \times n$ matrices. Can you find a formula for $\det \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$?

b) Show that number of up-crossings of a pattern is the same if the pattern is transposed and that therefore $\det(A^T) = \det(A)$.

4 Find the determinant of the Smith GCD matrix

$$A_{ij} = \mathbf{gcd(i,j)}, \text{ with } 1 \leq i, j \leq 5,$$

where $\gcd(a, b)$ is the greatest common divisor of a, b.   Remark:

One has explicit formulas for these so called **Smith matrices**, where 5 is replaced with an arbitrary

n. If you like a little experimental math research project, try to see whether you can guess a formula

for general n. Similar formulas are known if GCD is replaced by LCM all due to Smith from 1875.

Henry John Stephen Smith (1826-1883) was a remarkable mathematician. He is also known

for the Smith normal form of a matrix and that he discovered a Cantor like set. The Cantor set is

therefore now also called Smith-Volterra-Cantor set.

5 Use Laplace expansion to find a formula for the determinant of
the 5×5 matrix $L(5)$ which has 2 in the diagonal and 1 in the
side diagonals and 0 everywhere else. Find first the determinants
of $L(2), L(3), L(4)$ and then $L(5) = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. Can you
see how it continues? Optional: If you dare, try to prove your
conjecture.

Determinants I

The **determinant** of a $n \times n$ matrix A is defined as the sum $\Sigma_\pi (-1)^{|\pi|} A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)}$, where π is a permutation of $\{1, 2, \ldots, n\}$ and $|\pi|$ is the number of up-crossings in the pattern given by π. This is the Leibniz definition of determinants. By grouping the patterns according of the position in the first column, we get immediately the Laplace expansion $\det(A) = (-1)^{1+1} A_{11} \det(B_{i1}) + \cdots + (-1)^{1+n} A_{n1} \det(B_{n1})$, where for each entry a_{j1} in the first column form the $(n-1) \times (n-1)$ matrix B_{j1} in which the first column and the j'th row of A are deleted.