

Homework 11: Orthogonality

This homework is due on Monday, March 2, respectively on Tuesday, March 3, 2015. (Do this early as March 3 is exam day).

1 a) Find the angle between $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

b) What is the length of the vector $\begin{bmatrix} 1 \\ 2 \\ \vdots \\ 24 \end{bmatrix}$? Remark: This is known to be only vector of the form $[1, \dots, n]$ with $n > 1$ and integer length.

2 A vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ encodes data (x_1, \dots, x_n) . The **expecta-**

tion of x is the average $m = (x_1 + \dots + x_n)/n$. The vector

$X = \begin{bmatrix} x_1 - m \\ \vdots \\ x_n - m \end{bmatrix}$ is the **centered form** of x . Its expectation is

zero. If X, Y are the centered versions of x, y then $X \cdot Y$ is called the covariance of X and Y and $X \cdot X$ the variance of X and $|X|$ the standard deviation and the cosine of the angle between X and Y is the correlation coefficient. Statistics is just using different notation: they write $E[X]$, $\text{Var}[X]$, $\text{Cov}[X, Y]/n$, $\sigma(X)/\sqrt{n}$ for the expectation, variance, covariance and standard deviation. The correlation coefficient is the same. Yes, vectors are random variables.

We work with $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

- Find the expectations and standard deviations of \vec{x}, \vec{y} .
- Find the covariance and correlation coefficient between \vec{x}, \vec{y} .

3 If \vec{x}, \vec{y} are two vectors, we get data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ in the plane. The line $y = ax + b$ is called the **best linear fit**. We have $b = E[y] - aE[x]$, where $a = \text{Cov}[X, Y]/|X|^2$. Draw the 5 data points from problem **2** and find the best fit $y = ax + b$.

4 An orthogonal basis in R^n for which every vector has either entries -1 or 1 is called a **Walsh basis**. The corresponding matrix is a

Walsh matrix. Check that the columns of $W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} / 2$

form an orthonormal basis. Verify that one can get from W a 8×8 matrix encoding an orthonormal basis in R^8 by scaling

$A = \begin{bmatrix} W & W \\ W & -W \end{bmatrix}$ in the right way.

(Joseph Walsh graduated from Harvard in 1916, taught here from 1935-1966. Here is an open problem: nobody knows whether there is an orthogonal basis of vectors with entries $-1, 1$ with length $n = 668$. The corresponding matrices are called Hadamard matrices. The Walsh matrices above allow to construct examples for $n = 2^m$.)

5 Use an orthogonal basis of $x + y + z = 0$ to find the matrix of the projection onto that plane.

Orthogonality

Two vectors are **orthogonal** if $\vec{v} \cdot \vec{w} = 0$. The dot product of two vectors $\vec{v} = [v_1, v_2, \dots, v_n]$ and $\vec{w} = [w_1, w_2, \dots, w_n]$ is $\vec{v} \cdot \vec{w} = v_1w_1 + \dots + v_nw_n = 0$. The

length of a vector is $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$. The vector $\vec{v}/|\vec{v}|$ is called a **unit vector**. The Cauchy-Schwarz inequality $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$ allows to define the angle α by $\cos(\alpha) = (\vec{v} \cdot \vec{w})/(|\vec{v}| \cdot |\vec{w}|)$. The number $\cos(\alpha)$ is called the **correlation coefficient**. If it is positive, the vectors are **positively correlated**, if it is negative they are **negatively correlated**. Orthogonal vectors are **uncorrelated**. A basis is an orthonormal basis, if all vectors are perpendicular and have length 1. If they are just orthogonal one talks about an orthogonal basis. If we have an orthonormal basis of V , and Q be the matrix containing the basis vectors as column vectors. then the projection onto the space V is given by the matrix $P = QQ^T$, where Q^T is the transpose matrix.