Homework 8: Basis

This homework is due on Monday, February 23, respectively on Tuesday, February 24, 2015.

1 Which of following sets are linear spaces?
a) $W = \{(x, y, z) \mid x + y + z = 2\}$
b) $W = \{(x, y, z, w) \mid x = w, y = z\}$
c) $W = \{(x, x, x) \mid x \text{ real }\}$
d) $W = \{(x + 1, x + 2, x + 3) \mid x \text{ real }\}$
e) $W = \{(x, y, z) \mid x^2 + y^2 = 0\}$
f) $W = \{(x, y, z) \mid x^2 = 0\}$

2 Let V, W be linear subspaces of R^3. True or False? Give a reason.
a) The union of V and W is a linear subspace.
b) The intersection of V and W is a linear subspace.
c) $V^\perp$, the set of vectors perpendicular to V, is a linear subspace.
d) the intersection of V and the unit sphere is a linear subspace.

3 Check whether the given set of vectors is linearly independent

a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} \right\}.$

b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$

4 Find a basis for the image as well as as a basis for the kernel of the following matrices

a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

5 The orthogonal complement of a subspace V of R^n is the set $V^\perp$ of all vectors in R^n that are perpendicular to every single vector in V. Find a basis for the orthogonal complement of each of the following:

a) The line L in R^3 spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

b) The plane Σ in R^4 spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

Basis

A set of vectors is a **linear space** if it contains the zero vector, is closed under addition and scalar multiplication. The kernel and image of a matrix are examples of linear spaces. If V, W are linear spaces and $V \subset W$, we say V is a subspace of W. The vectors $v_1, ..., v_n$ form a **basis** of a linear space if it **spans** the space and if the vectors are linearly independent. Spanning means that every vector v can be written as a linear combination of vectors in v_k, linear independence means that $a_1 v_1 + ... a_n v_n = 0$ has only the solution $a_1 = ... a_n = 0$. An important example of a basis in R^n is $\{e_1, ..., e_n\}$. Use row reduction to find out whether vectors are linearly independent. Place the vectors v_i as columns in a matrix A then row reduce A. If every column has a leading 1, then the columns form a basis for the image of A. The column vectors are linearly independent if A has a trivial kernel.