

## Homework 7: Image and Kernel

This homework is due on Friday, February 20, respectively on Tuesday, February 24, 2015.

- 1 Find the kernel of the transformation  $x \rightarrow Ax$ , then write down a set of vectors which span the image of  $A$ .

a)  $\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 3 & 4 & 3 & 2 \\ 3 & 4 & 5 & 4 & 3 \\ 4 & 5 & 6 & 5 & 4 \end{bmatrix}$ , b)  $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ , c)  $[1 \ 2 \ 3 \ 4]$ .

d)  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ .

- 2 a) Give an example of a transformation from  $R^6$  to  $R^4$  for which

the image is a plane spanned by the two vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ .

b) Express the kernel of the  $1 \times 3$  matrix  $A = [1 \ 2 \ 3]$  as the image of a  $3 \times 2$  matrix  $B$ .

- 3 a) What is the image and kernel of the rotation–dilation  $\begin{bmatrix} 5 & 12 \\ -12 & 5 \end{bmatrix}$ ?

b) What is the image and kernel of the projection  $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ ?

c) What is the image and kernel of the reflection  $\frac{1}{13} \begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$ ?

d) What is the image and kernel of the shear  $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ ?

e) What is the image and kernel of the matrix  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ .

4 Let  $A$  be an arbitrary matrix and  $B = \text{rref}(A)$ .

a) Is it true that  $\ker(A) = \ker(B)$ ? Explain why or why not.

b) Is it true that  $\text{im}(A) = \text{im}(B)$ ? Explain why or why not.

c) Be creative and find a  $2 \times 2$  matrix for which  $\text{im}(A) = \ker(A)$ .

5 Let  $A$  be a  $n \times n$  matrix.

a) What is the relation between the kernel of  $A$  and the kernel of  $A^2$ ?

b) What is the relation between the image of  $A$  and the image of  $A^2$ ?

c) What is the relation between the kernel of  $A$  and the kernel of  $A^2 + A$ ?

d) What is the relation between the image of  $A$  and the image of  $A^2 + A$ ?

## Image and kernel

The **image** of a linear map  $x \rightarrow Ax$  is the set of all vectors  $Ax$ . The **kernel** is the set of vectors  $x$  which satisfy  $Ax = 0$ . The columns of  $A$  **span** the image of  $A$ . Every  $x \in \text{im}(A)$  can be written as a linear combination of column vectors. The image and kernel are both linear spaces: they are closed under addition, scalar multiplication and contain the zero vector. The kernel of a  $n \times n$  matrix is  $\{0\}$  if and only if  $A$  is invertible if and only if the image is  $\mathbb{R}^n$  if and only if  $\text{rref}(A) = 1$ .