Homework 3: The number of solutions

This homework is due on Friday, February 6, respectively on Tuesday February 10, 2015.

1 Given $A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 10 \end{bmatrix}$. In each of the vectors $\vec{b}$ given, determine

whether the system $A\vec{x} = \vec{b}$ has $0, 1$ or ∞ many solutions.

a) $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ b) $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ c) $\vec{b} = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$ d) $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ e) $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

2 Consider the set X of all 2×2 matrices with matrix entries 0 or 1. The probability of a set of matrices Y with some property is the number of matrices in Y divided by the number of matrices in X.
a) What is the probability that the rank of the matrix is 1?
b) What is the probability that the rank of the matrix is 0?
c) What is the probability that the rank of the matrix is 2?

3 As in the previous problem, now also the 2-vector b take randomly the values 0,1, we can look at all the possible equations $Ax = b$, where A, b are obtained with 0 or 1 entries. The probability space has now 64 elements.
a) What is the probability that the system has a unique solution?
b) What is the probability that the system has no solution?
c) What is the probability that the system has infinitely many solutions?

4 Build your own system of equations for three variables:
  a) An example with exactly one solution.
  b) An example with no solutions.
  c) An example where the solution is a plane.
  d) An example where the solution is a line.
  e) An example where the solution space is three dimensional.

5 In a herb garden, the soil has the property that at any given point the humidity is the sum of the neighboring humidities. Samples are taken on a hexagonal grid on 14 spots. The humidity at the four locations x, y, z, w is unknown. Solve the equations
$$\begin{vmatrix} x = & y+z+w+2 \\ y = & x+w\text{-}3 \\ z = & x+w\text{-}1 \\ w = & x+y+z\text{-}2 \end{vmatrix} \text{ using}$$
row reduction.
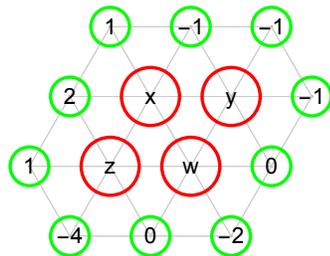
Main properties

A system which has a solution is called **consistent**. Otherwise it is called **inconsistent**.

We have a unique solution to $A\vec{x} = \vec{b}$ if and only if $\text{rref}(A)$ has a leading 1 in every column and the system is consistent. We have no solution if and only if $\text{rref}(A|b)$ has a leading 1 in the last column. In all other cases we have infinitely many solutions.