Math21b

Final Review Spring 2015

selected slides

Oliver Knill, May 4, 2015

Matrices

n x m matrix
n rows and m columns

Columns:
image of basis vectors

$$(AB)^{-1}=B^{-1}A^{-1}$$

$$(AB)^{T}=B^{T}A^{T}$$

$$AB \neq BA$$

in general

$$B = S^{-1} A \, S$$

similarity

$A x = b$

$x = A^{-1} b$

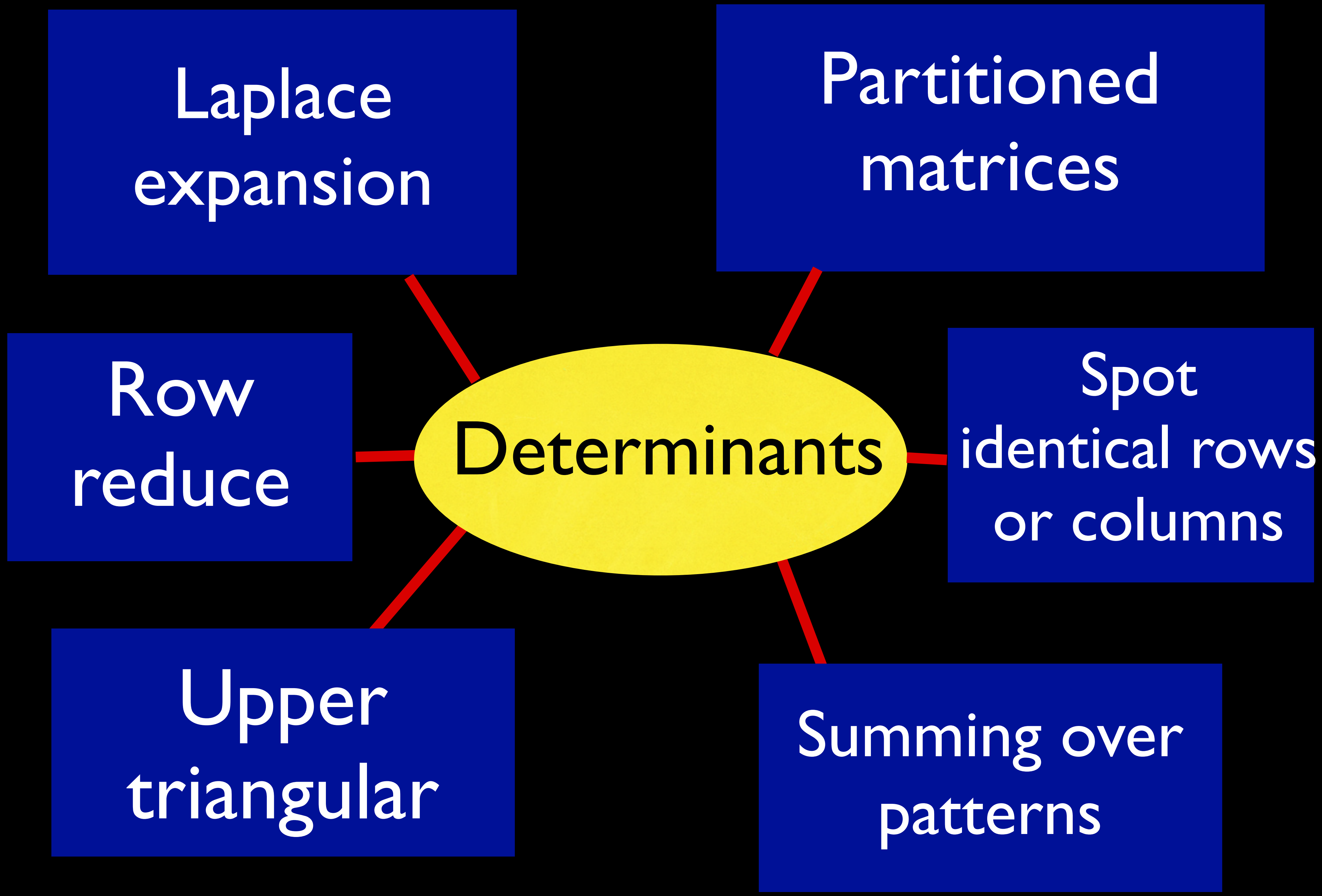
row reduce
$[A|\ b]$

Linear
equations

Solutions
are $x_0 + \ker(A)$

Least square
solution

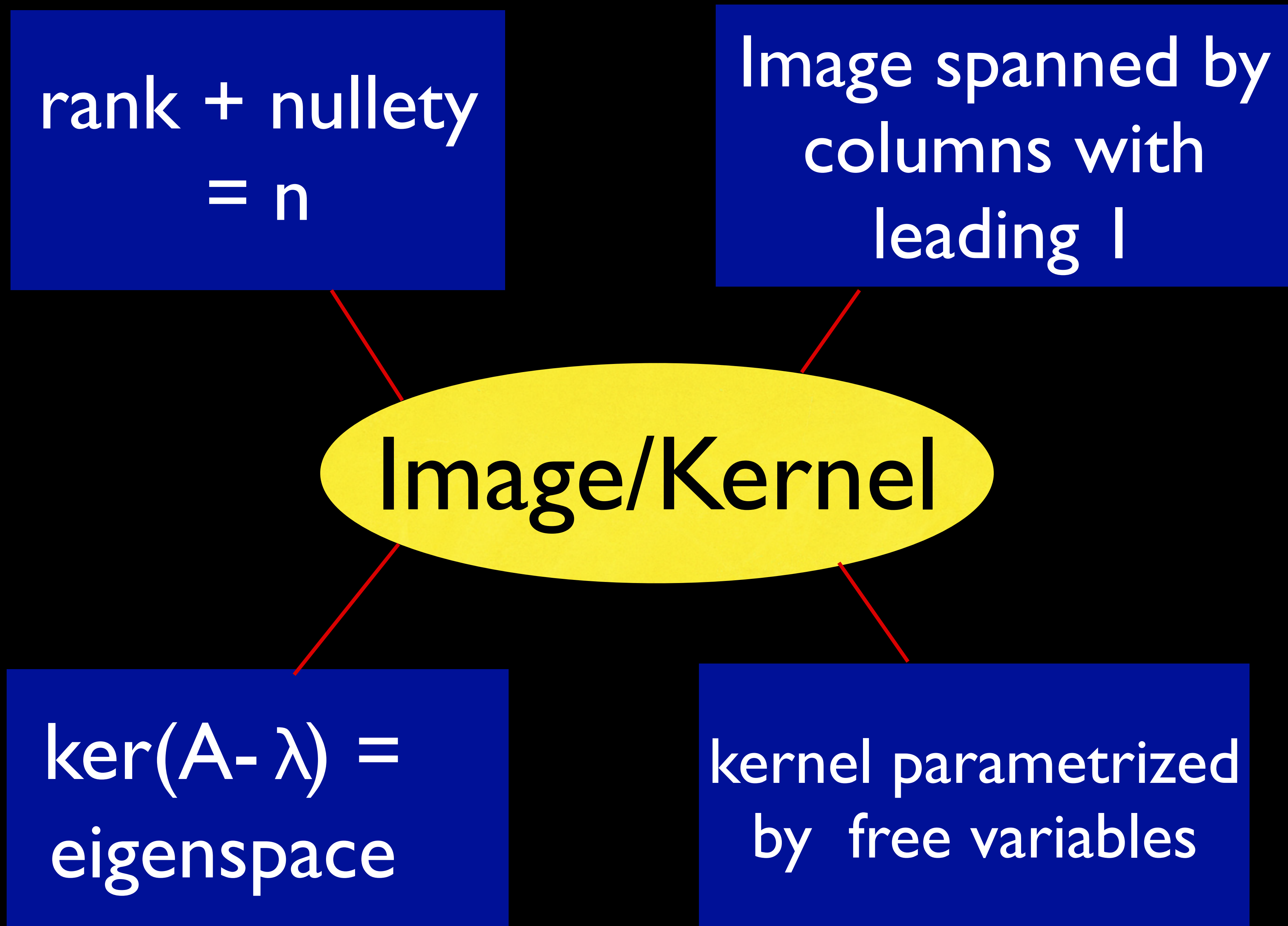
consistent:
have solution

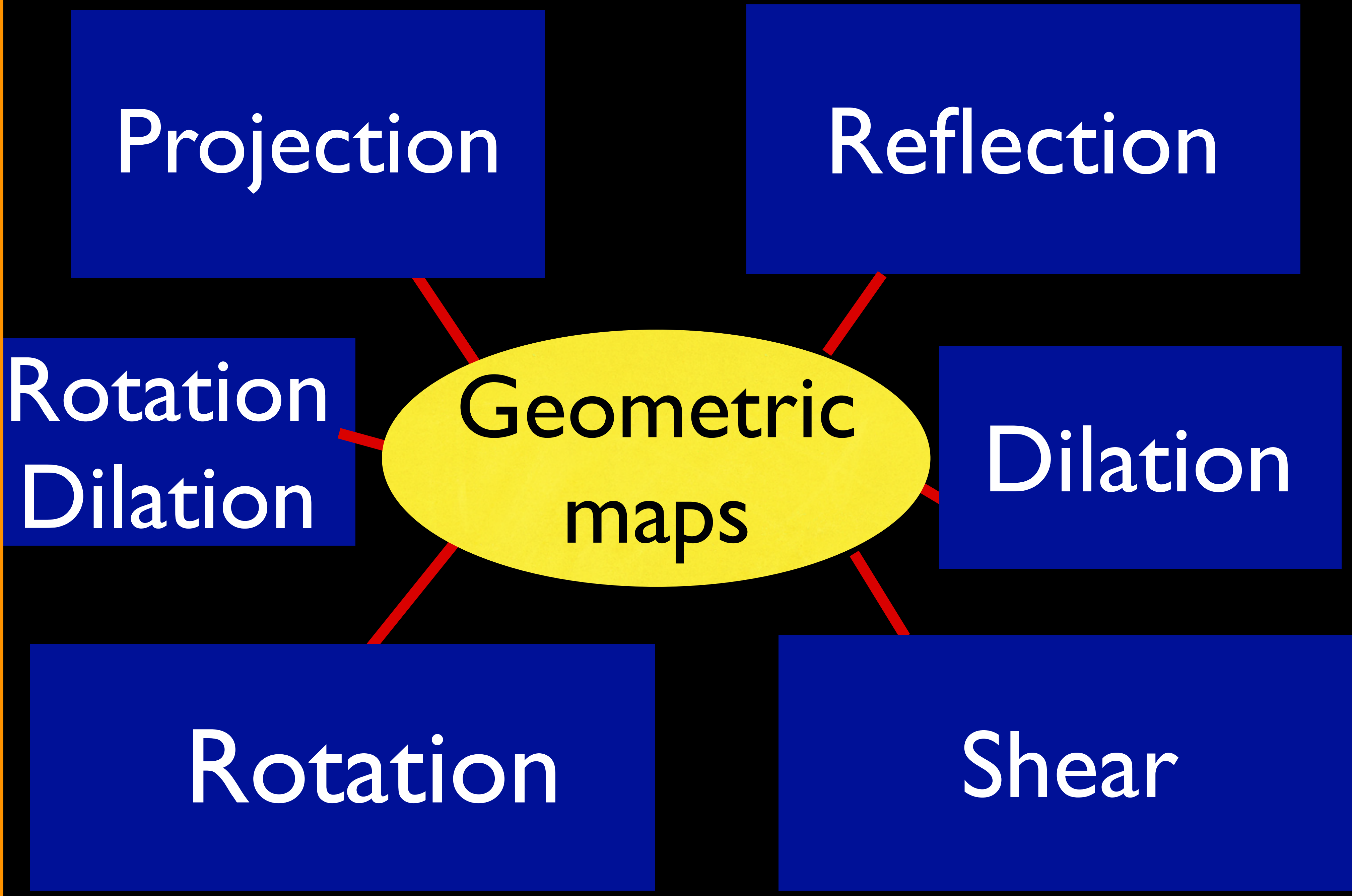

Determinants
Laplace expansion
Partitioned matrices
Row reduce
Spot identical rows or columns
Upper triangular
Summing over patterns

$$\det\begin{pmatrix} 1 & 2 & 3 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 9 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 9 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 & 15 & 16 \end{pmatrix} = ?$$

$$\det \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = ?$$

$$\det \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = ?$$


Image/Kernel
rank + nullety = n
Image spanned by columns with leading 1
ker(A- λ) = eigenspace
kernel parametrized by free variables


Projection
Reflection
Rotation
Dilation
Geometric
maps
Dilation
Rotation
Shear

Projection

AA^T

if columns orthogonal

$A(A^TA)^{-1}A^T$

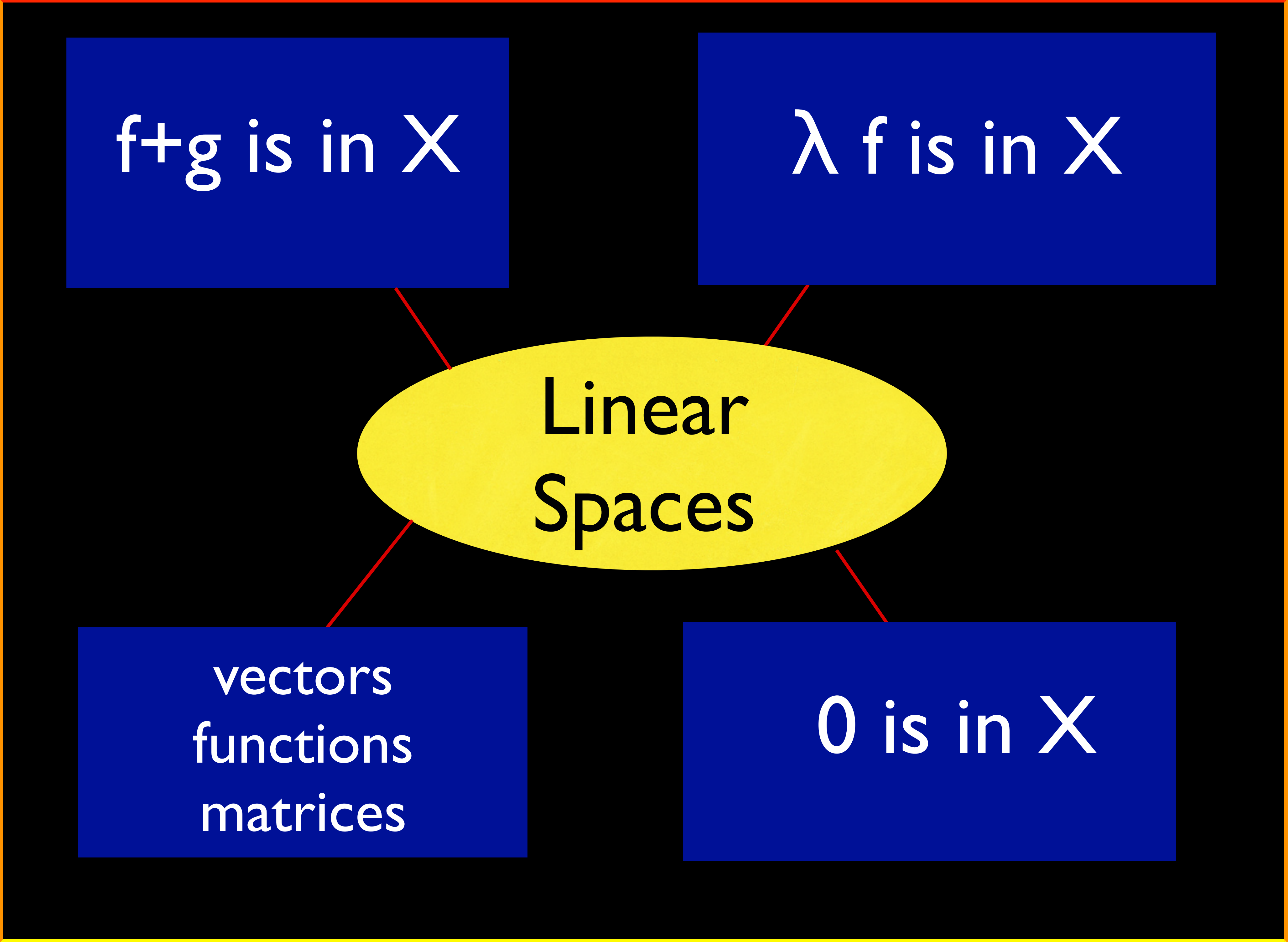
$x = (A^TA)^{-1}A^T$

least square solution

$P^2 = P$

$Pv = (u.v)\,v$

onto one dimensional line


f+g is in X
λ f is in X
Linear
Spaces
vectors
functions
matrices
0 is in X

Linear Maps

$T(f)$ is in X

$T(0) = 0$

$T(f+g) = T(f) + T(g)$

$T(\lambda f) = \lambda T(f)$

$$x' = \lambda x$$

Solution:

$$x(t) = x(0)\, e^{\lambda t}$$

THE MOTHER OF ODE'S

$$x'' = -c^2 x$$

Solution:

$$x(t) = x(0)\cos(c\,t) + x'(0)\sin(c\,t)/c$$

THE FATHER OF ODE'S

Cookbook

Solution of $p(D)\, f = g$

Solve the homogeneous problem

$$p(D)\, f = 0$$

Find a special solution

How do we guess the special solution?

right hand side	Try with
$\sin(kt)$	$A\sin(kt) + B\cos(kt)$
e^{kt}	Ae^{kt}
1	A
t	$At + B$
t^2	$At^2+Bt + C$
$1+\sin(t)$	$C+A\sin(t)+B\cos(t)$
If in Kernel or double roots multiply with t.	$At\sin(kt) + Bt\cos(kt)$

Nonlinear systems

We look at equations in the plane

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

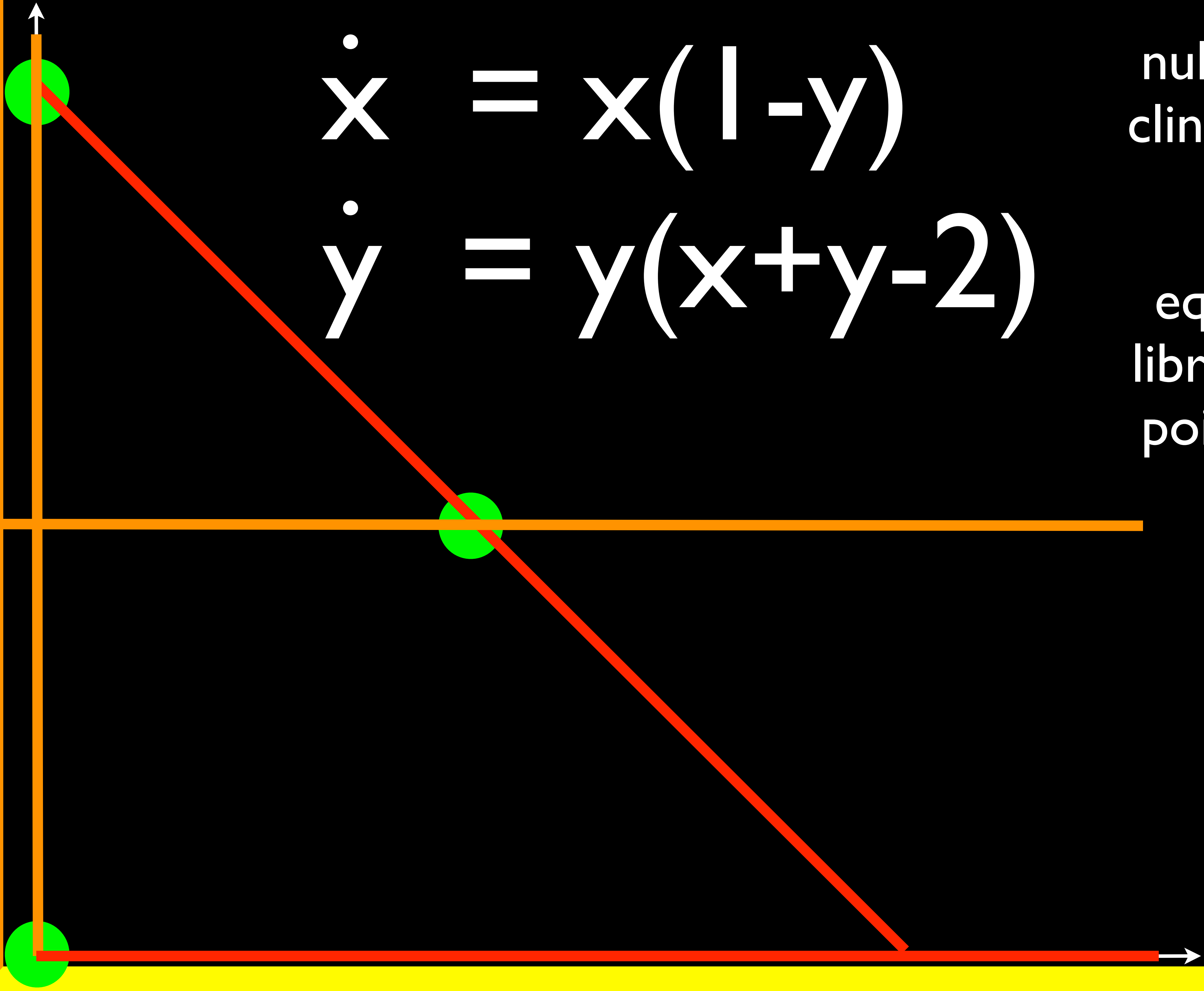
An example

$$\dot{x} = x(1-y)$$

$$\dot{y} = y(x+y-2)$$

$$\dot{x} = x(1-y)$$

$$\dot{y} = y(x+y-2)$$

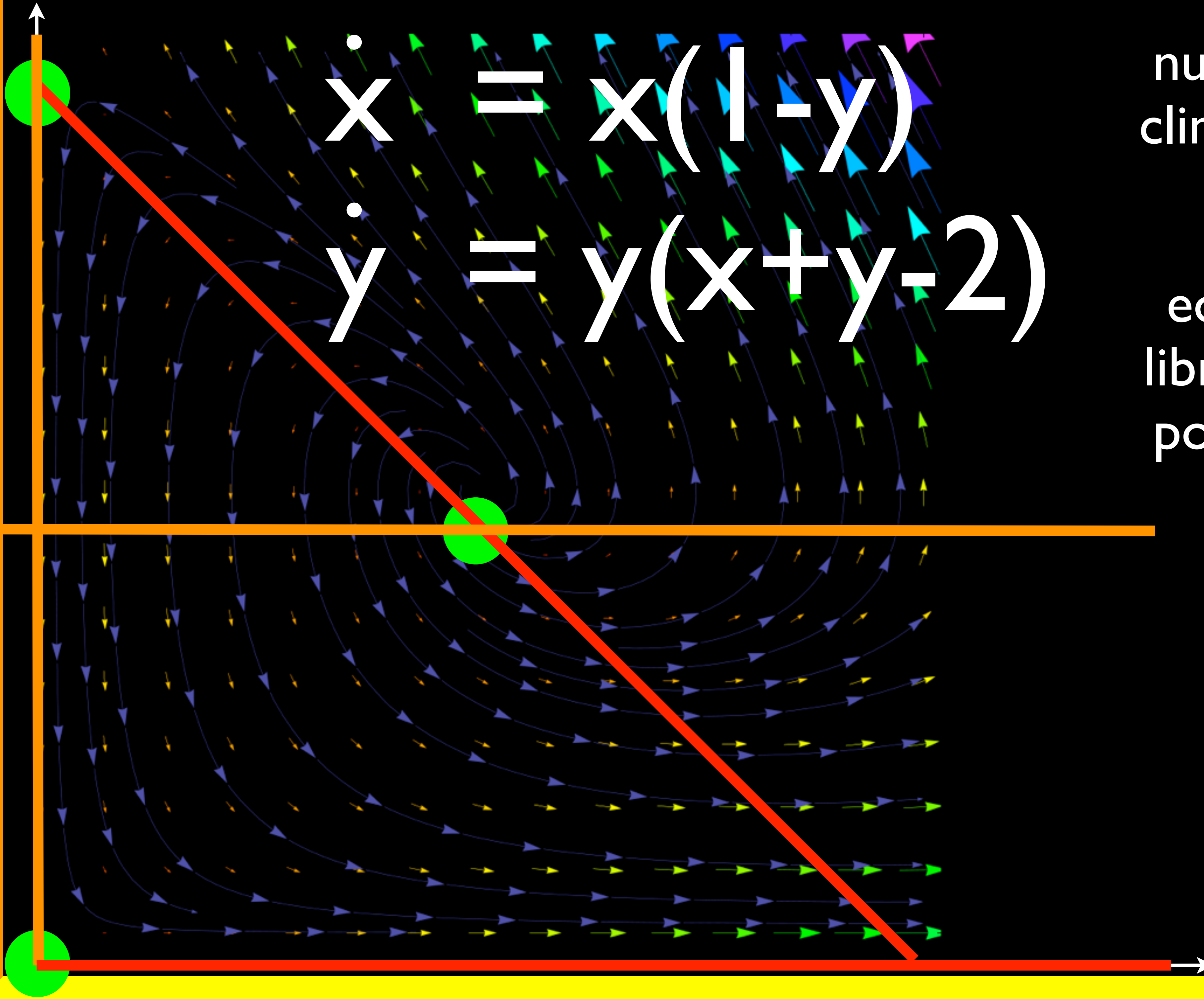


null-clines

equi-librium points

$$\dot{x} = x(1-y)$$

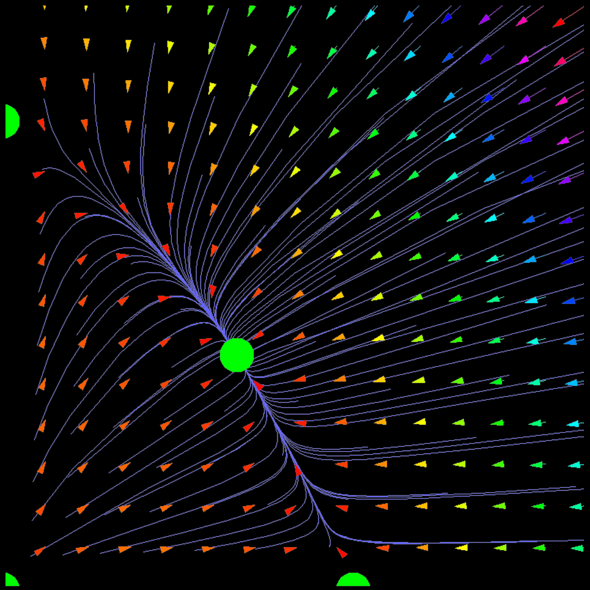
$$\dot{y} = y(x+y-2)$$



null-clines

equi-librium points

Jacobean matrix

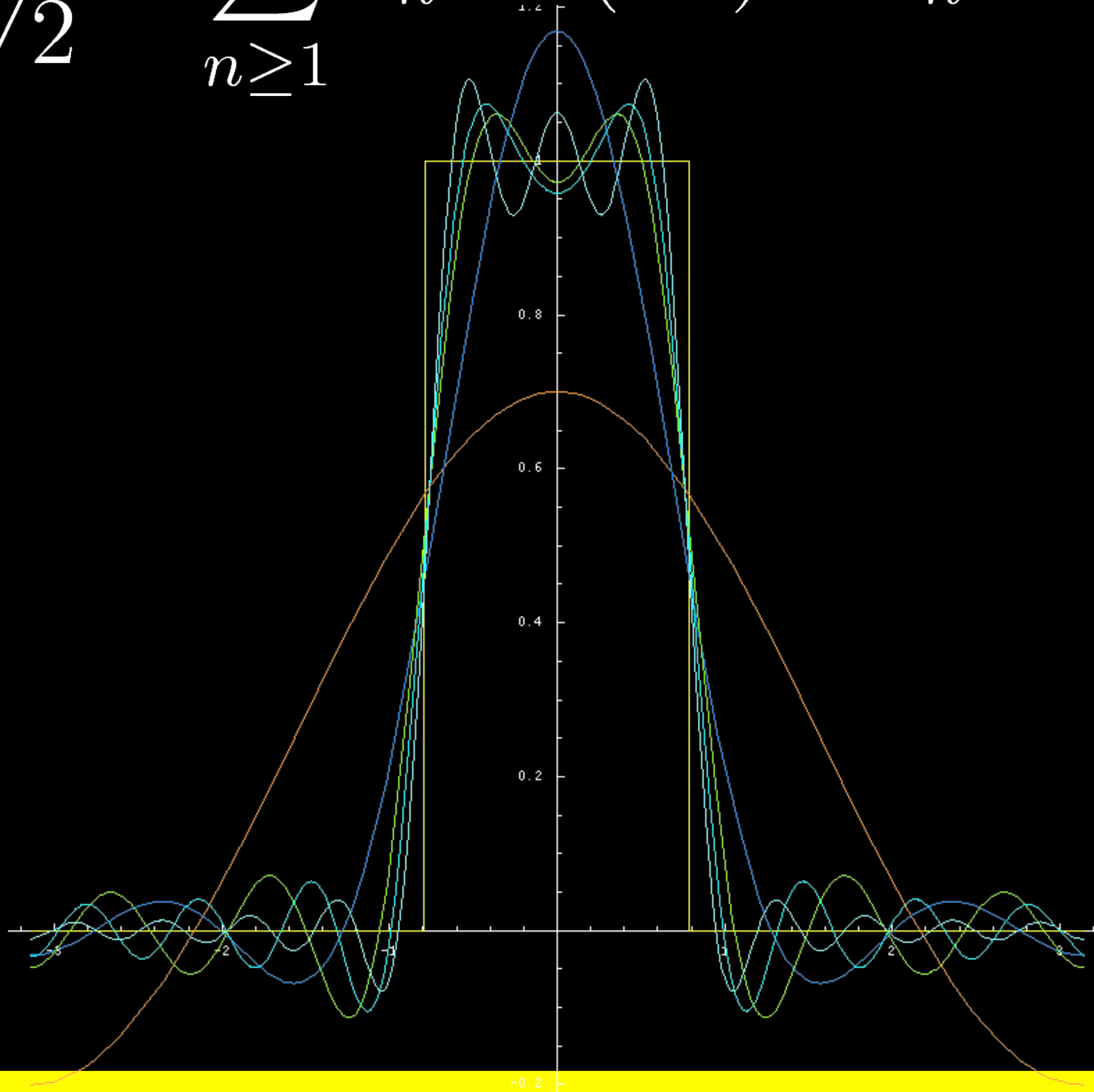


$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Fourier analysis

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$$



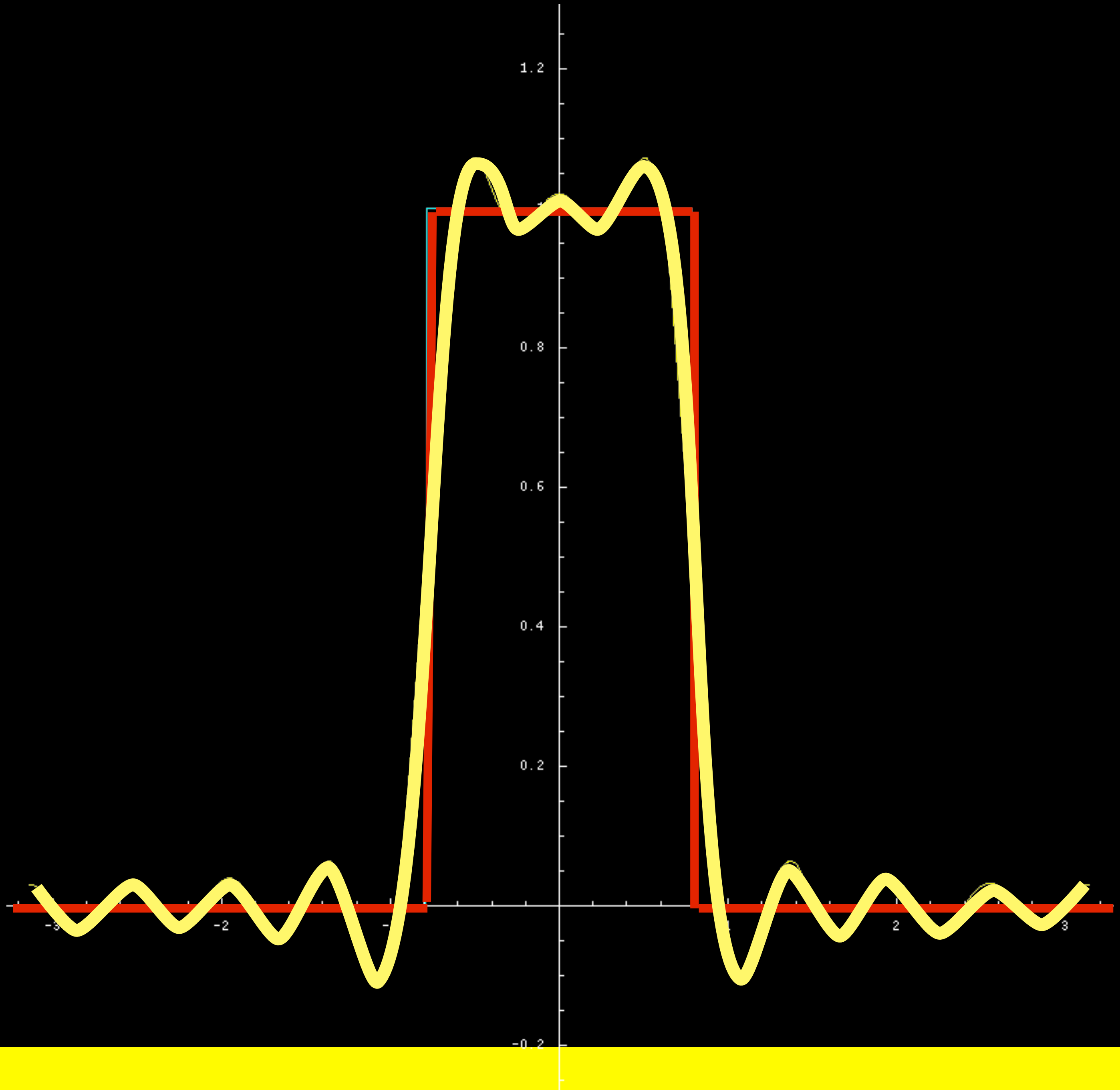
Fourier coefficients:

$$a_0 = \frac{1}{\pi}\int_{\pi}^{\pi} f(x)\,\frac{1}{\sqrt{2}}\,dx$$

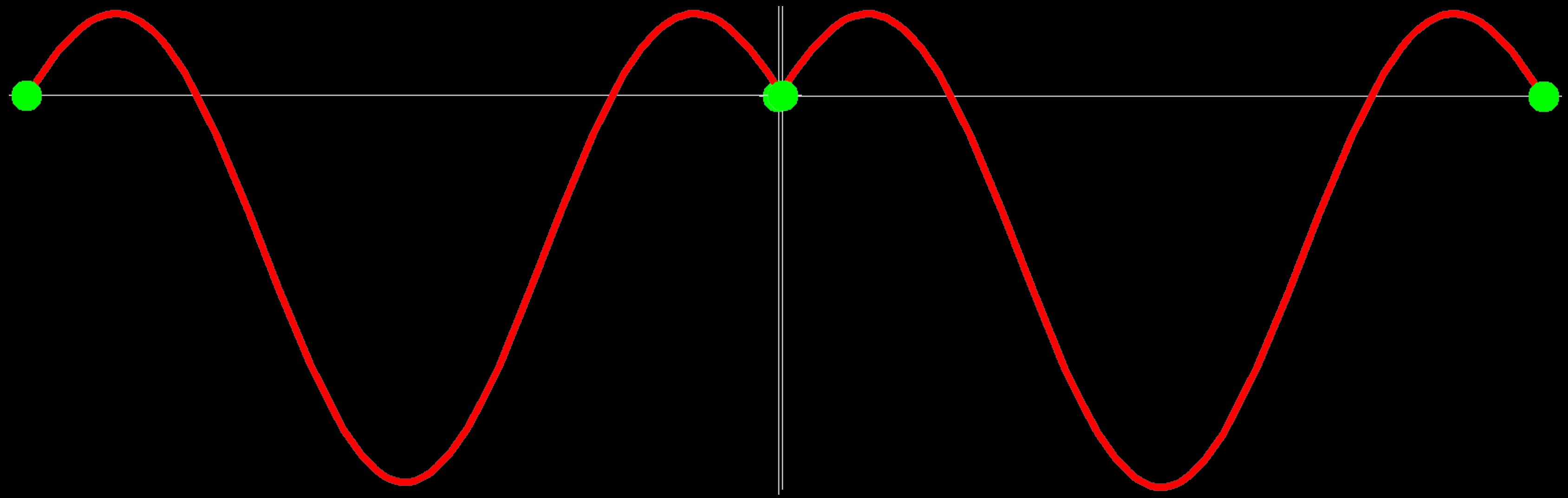
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Fourier approximation

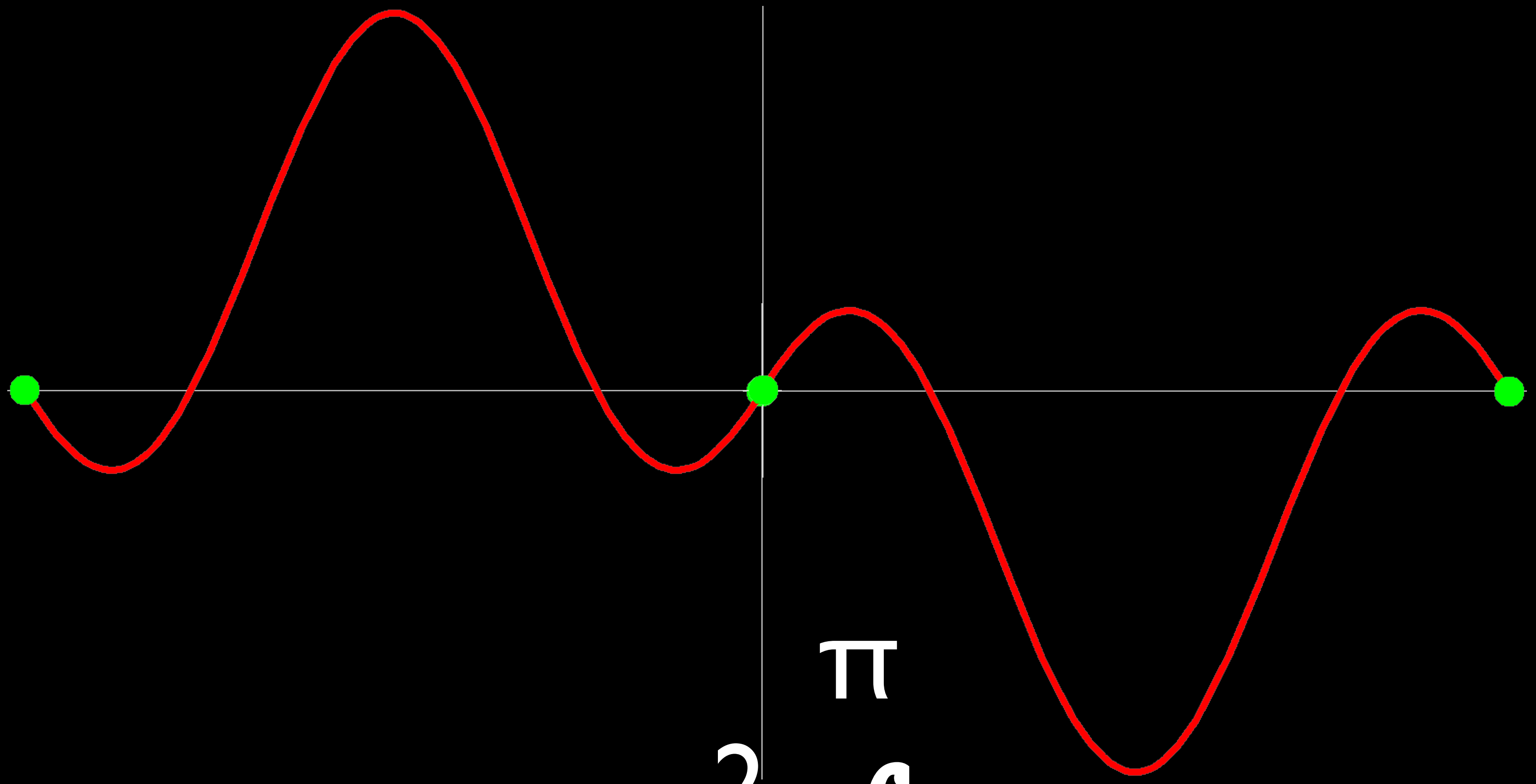


Even Functions



$$\text{cos- series} \quad \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

Odd Functions

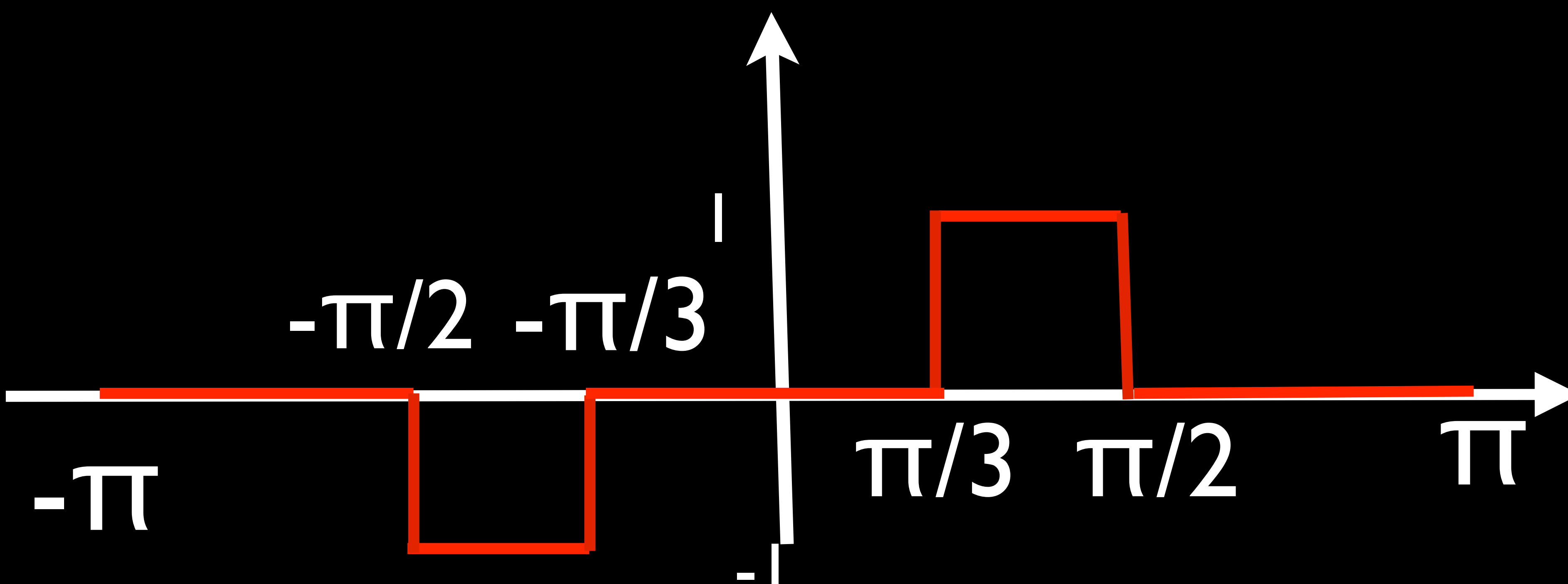


sin- series $\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

Blackboard Problem

Find the Fourier series of the following function:

f (x,0) =



Parseval Identity

$$||f||^2 = a_0^2 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2$$

Marc-Antoine Parseval

Partial differential equations

heat equation: $u_t = u_{xx}$

heat type equation: $u_t = p(D^2)\, u$

wave equation: $u_{tt} = u_{xx}$

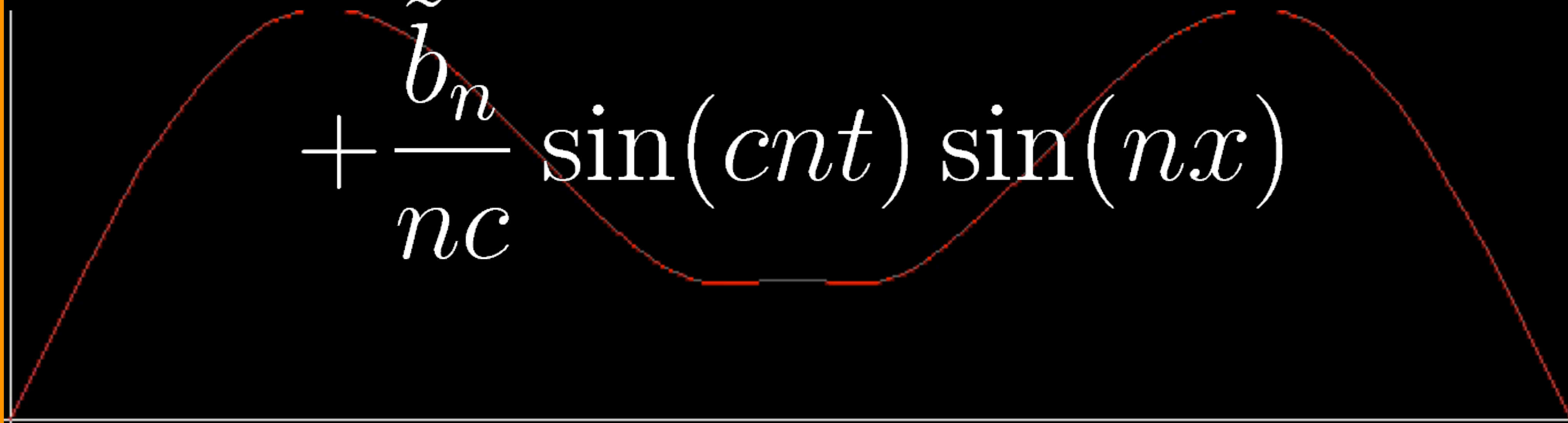
wave type equation: $u_{tt} = p(D^2)\, u$

Heat evolution

$$f(x,t) = \sum_{n \geq 1} b_n e^{-n^2 \mu t} \sin(nx)$$

b_n are Fourier coefficients of f(x,0)

$$f(x,t) = \sum_{n \geq 1} b_n \cos(cnt) \sin(nx) + \frac{\tilde{b}_n}{nc} \sin(cnt) \sin(nx)$$



b_n are Fourier coefficients of f(x,0)

$\tilde{b}_n$ are Fourier coefficients of f'(x,0)

Wave evolution

Heat Problem 1

$$u_t = 9u_{xx} - 2u$$

$u(x,0)=\sin(3x)$

Wave Problem 1

$$u_{tt} = 9u_{xx} - 2u$$

$u(x,0) = \sin(3x)$

$u_t(x,0) = 0$

Wave Problem II

$$u_{tt} = 9u_{xx} - 2u$$

$u(x,0)=0$

$u_t(x,0)=\sin(7x)$