MWF9 George Boxer
MWF10 Omar Antolin
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TThu11 Arul Shankar

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F The matrix $\begin{bmatrix} 2 & -c \\ c & 3 \end{bmatrix}$ is always invertible for $c \in \mathbb{R}$.

2) T F The solutions of $f''' + f'' + 17f = e^t$ form a linear subspace of $C^\infty(\mathbb{R})$

3) T F The solutions of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ form a linear subspace of $\mathbb{R}^2$.

4) T F If A and B are 5×5 matrices, then $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$.

5) T F Similar matrices have the same rank.

6) T F If A is a matrix which has orthonormal columns, then $\det(AA^T) = \det(A^TA)$.

7) T F If A is a 2×2 matrix with $\det(A) < 1$, then the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ has a stable origin.

8) T F If A, B are $n \times n$ matrices, then $\det(2A + 3B) = 2^n \det(A) + 3^n \det(B)$.

9) T F If A and B are 2×2 matrices with the same trace and the same determinant, then A and B have the same eigenvalues.

10) T F If $A = QR$ is the QR decomposition obtained by Gram-Schmidt orthogonalization, then A and R have the same eigenvalues.

11) T F If $\vec{x}^*$ is the least-squares solution of $A\vec{x} = \vec{b}$ then $\vec{b}$ is orthogonal to $A\vec{x}^*$.

12) T F If two matrices are symmetric and have the same eigenvalues (with the same algebraic multiplicities), then they are similar.

13) T F Every system of linear equations has a least square solution.

14) T F If a real 2×2 matrix A has i as an eigenvalue, it is orthogonal.

15) T F If $A = BCD$, where A, B, C, D are all 3×3 matrices and A is not invertible, then one of the matrices B, C, D are not invertible.

16) T F If every eigenvalue λ of a matrix A satisfies $\text{Re}(\lambda) < 1$, then $\vec{0}$ is an asymptotically stable equilibrium of the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$.

17) T F There is an $n \times n$ matrix A which has an eigenvalue λ of geometric multiplicity 0.

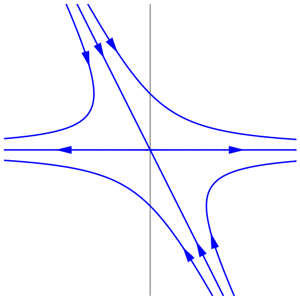
18) T F If $f(x) = 3\cos(7x) + 4\sin(2004x) + 2$, then $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \, dx = 33$.

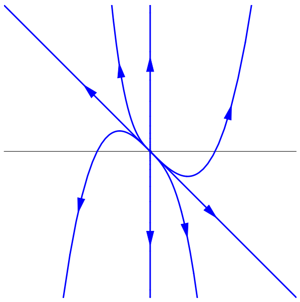
19) T F 7 is an eigenvalue of $T(f) = f'' + 7f' + 77f$ on the space $X = C^\infty(\mathbb{R})$ of smooth functions on the real line $\mathbb{R}$.

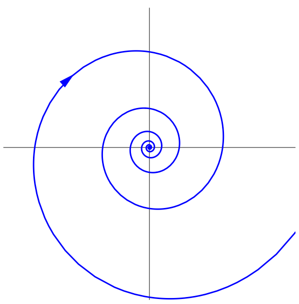
20) T F If $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$ and $\vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ then $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

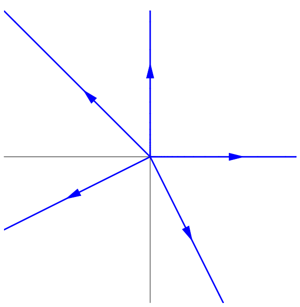
Problem 2) (10 points)

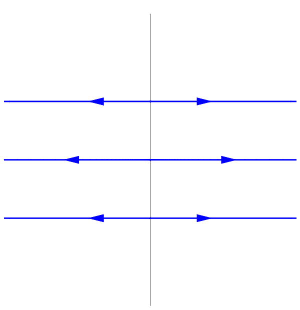
Pick the five of the dynamical system 1) - 9) which correspond to the phase portraits.











1. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$

2. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \vec{x}$

3. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \vec{x}$

4. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}$

5. $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \vec{x}$

6. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$

7. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}$

8. $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 7 \\ -7 & 1 \end{bmatrix} \vec{x}$

9. $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & 7 \\ -7 & -1 \end{bmatrix} \vec{x}$

Problem 3) (10 points)

To match the dynamical systems to the left with the description to the right, fill in a)-e) in the boxes. No justifications are necessary.

a) $\frac{d}{dt}x = \sin(xy), \frac{d}{dt}y = x^2 + y$

b) $f_t = f_{xxxx}$

c) $\vec{x}(t+1) = A\vec{x}(t)$

d) $\frac{d}{dt}\vec{x} = A\vec{x}$

e) $f'' + f' + f = \sin(t)$

	Partial differential equation
	Linear system of ordinary differential equations
	nonlinear differential equation
	inhomogeneous linear differential equation
	discrete dynamical system.

To match the matrices to the left with the description to the right, distribute a)-e) in the boxes.

a) $\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$

b) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

c) $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$

	skew symmetric matrix
	rotation
	reflection
	projection
	shear

Problem 4) (10 points)

Let $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 2 & 5 \end{bmatrix}$.

a) (3 points) Find all eigenvalues of A with their algebraic multiplicities.

b) (3 points) Find the geometric multiplicities of each eigenvalue.

c) (2 points) Is A diagonalizable?

d) (2 points) What is the determinant of A^3?

Problem 5) (10 points)

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

a) Find A^{-1}.

b) Solve $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

c) Find the matrix of $T(\vec{x}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}$ with respect to the basis $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$.

Problem 6) (10 points)

You have only to solve 5 from the following 6 problems to have full credit. But you can attempt all of them.

a) (2 points) Find a 3×3 matrix A of rank 1 with no zero entries.

b) (2 points) Find a matrix A which has $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the image of A.

c) (2 points) Find a 3×3 matrix A whose kernel is 2-dimensional.

d) (2 points) Find a 2×2 matrix A with different eigenvalues such that $A^2 - 3A + 2I_2$ is the zero matrix.

e) (2 points) Find a 2×2 matrix A for which A^{-1} and A^T have the same eigenvectors.

f) (2 points) Find a 3×3 matrix A such that every vector in $\mathbb{R}^3$ is an eigenvector of A with eigenvalue 3.

Problem 7) (10 points)

Let $A = \begin{bmatrix} 1 & 2 & -3 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -4 \end{bmatrix}$.

a) (4 points) Find a basis of $\ker(A)$.

b) (2 points) Find the rank of A.

c) (1 point) Is there a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution?

d) (1 point) Is there a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has exactly one solution?

e) (1 point) Is there a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has infinitely many solutions?

f) (1 point) Find $\det(A)$.

Problem 8) (10 points)

Let V be the plane spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Find the matrix of reflection at the plane V.

Problem 9) (10 points)

a) (4 points) Find all the solutions of the differential equation $f' + 2f = e^{-2t}$.

b) (4 points) Find all the solutions of the differential equation $f'' + 4f' + 4f = e^{-2t}$.

c) (2 points) Find the kernel of $T(f) = f'' + 4f' + 4f$.

Problem 10) (10 points)

We analyze the nonlinear dynamical system

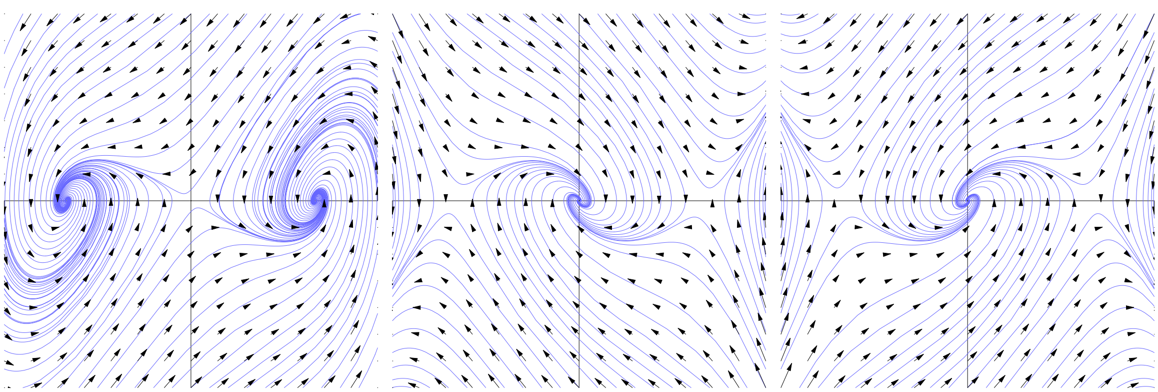
$$\begin{aligned} \frac{d}{dt}x &= y \\ \frac{d}{dt}y &= x^3 - x - y \end{aligned}$$

a) (2 points) Draw the nullclines, and indicate the direction of the field along the nullclines and inside the regions determined by the nullclines.

b) (2 points) Find all the equilibrium points.

c) (4 points) Analyze the stability of all the equilibrium points

d) (2 points) Which of the phase portraits A,B,C below belong to the above system?



A B C

Problem 11) (10 points)

Find the Fourier series of the function $f(x) = \cos(x) + \sin(2x) + x$ defined on $[-\pi, \pi]$. Show all computation steps.

Problem 12) (10 points)

a) Find the solution of the heat equation $f_t = 3f_{xx}$ for which $f(x,0) = \sin(x) + \frac{1}{2}\sin(10x)$.

b) Find the solution of the wave equation $f_{tt} = 400f_{xx}$ for which $f(x,0) = \sin(x) + \frac{1}{2}\sin(10x)$ and $f_t(x,0) = \sin(20x)$.

Problem 13) (10 points)

A vibrating string with friction is modeled by the **driven wave equation**

$$u_{tt} = u_{xx} - u \ .$$

Find the solution of this equation if $u(x,0) = 3\sin(5x)$ and initial speed $u_t(x,0) = \sin(11x)$. As usual, we work with $x \in [0,\pi]$.

Problem 14) (10 points)

If D is the differentiation operator $Df(x) = f'(x)$, we can define $e^D f(x) = \sum_{n=0}^{\infty} \frac{D^n f}{n!}$.

a) Verify that $e^{Dt}f(x)$ satisfies the transport equation $\frac{d}{dt}f = Df$.

b) Also $f(x+t)$ is a solution to the transport equation $f_t = Df$. It must therefore be the same as the solution $e^{Dt}f$ obtained in a). We have derived the equation

$$f(x+t) = e^{Dt}f \ .$$

It is known the name Taylor formula.