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- Start by writing your name in the above box and check your section in the box to the left.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or un-staple the packet.

- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) True or False? No justifications are needed.

1) T F The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ is symmetric.

2) T F The vector $\vec{v} = 0$ is always in the kernel of a matrix A.

3) T F For any 2×2 matrices A, the kernel of A is a subspace of the image of A.

4) T F If a 2×2 matrix A is diagonalizable then the matrix A^T is diagonalizable.

5) T F The Fourier series of the function $f(x) = \sin^2(x)$ only involves cosine terms and the constant term.

6) T F The space of smooth functions satisfying the equation $f(x) = f(-x)^2$ forms a linear space.

7) T F The algebraic multiplicity of an eigenvalue 0 of A is equal to the nullity of A.

8) T F The sum $A + B$ of two real matrices A, B which both have eigenvalues $\lambda = i, -i$ is a matrix with eigenvalues $i, -i$.

9) T F A discrete dynamical system $x(t + 1) = Ax(t)$ with a 2×2 matrix A is asymptotically stable if $\det(A) < 0$.

10) T F If $x'(t) = Ax(t)$ and $x(t + 1) = Ax(t)$ are both asymptotically stable, then all real eigenvalues λ satisfy $-1 < \lambda < 0$.

11) T F The sum of two orthogonal projections is an orthogonal projection.

12) T F The function $f(x, t) = e^{-6t} \sin(3x)$ solves the heat equation $f_t = 2f_{xx}$.

13) T F If a system of linear equations $A\vec{x} = \vec{c}$ with 2×2 matrix A has infinitely many solutions, then there exists $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution.

14) T F The equilibrium point $(0, 0)$ of the nonlinear system $x' = x^2$, $y' = y^2$ is asymptotically stable.

15) T F All real symmetric matrices are diagonalizable over the real numbers.

16) T F $||3\sin(5x) - 7\sin(10x)|| = \sqrt{5^2 + 10^2}$, where the length $||f||$ of f is defined by the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx$.

17) T F The sum of the eigenvalues of a symmetric 2×2 matrix is 0.

18) T F The differential equation $x''(t) + 9x(t) = \sin(3t)$ with $x(0) = 2, x'(0) = 2$ has solutions for which $|x(t)|$ becomes arbitrary large.

19) T F For any invertible matrix, A^{-1} is similar to A.

20) T F If an $n \times n$ matrix has the property that the sum of all matrix entries are zero, then the matrix is not invertible.

2

a) (6 points) We are given a real 3×3 matrix A and define $B = A^3 + A$.

Statement	Always true
If A is invertible, then B is invertible	
If A is diagonalizable, then B is diagonalizable	
If A is symmetric, then B is symmetric	
If A has a zero determinant, then B has a zero determinant	
If A has zero traces, then B has zero trace	
If A has an eigenvalue 1, then B has an eigenvalue 2	

b) (4 points) Match the differential equations with possible solution graphs.

Enter A-D	Differential equation
	$f''(t) + f(t) = 2\sin(2t)$
	$f'(t) = 2\sin(t)$
	$f'(t) - f(t) = 2\sin(2t)$
	$f''(t) + f(t) = 2\sin(t)$

A)

B)

C)
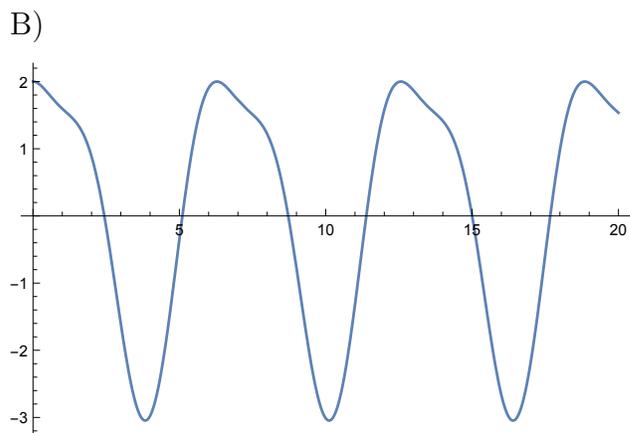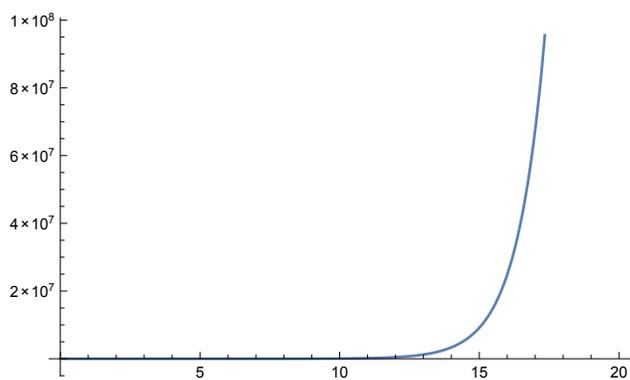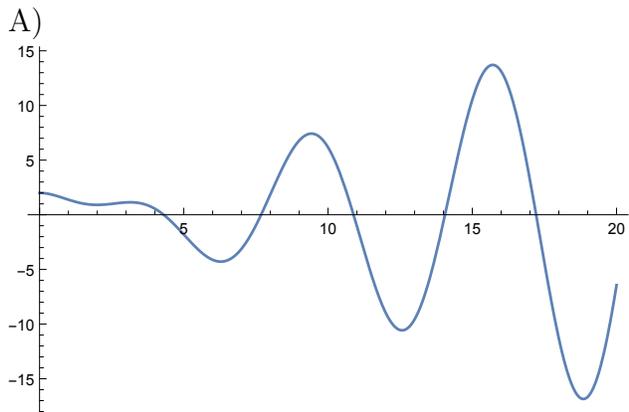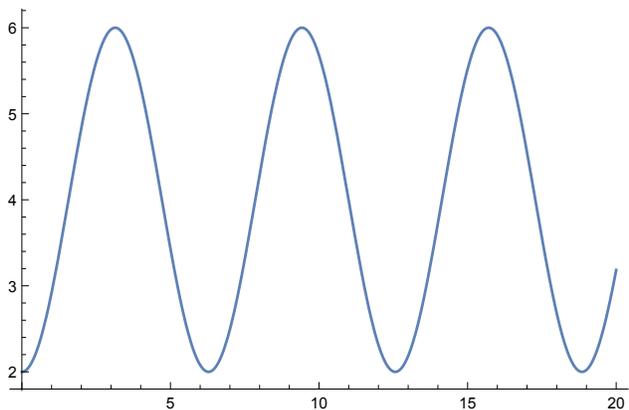
D)

a) (6 points) Assume T is a transformation on C^∞, the linear space of smooth functions on the real line. Which of the following transformations are linear?
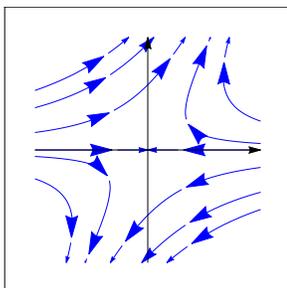
Transformation	Check if linear
$Tf(x) = 3f(5x^2)$	
$Tf(x) = f(4-x)$	
$Tf(x) = \sin(x)f'(x)$	
$Tf(x) = f(5)f(x)$	
$Tf(x) = f(5)x^2$	
$Tf(x) = 1 + f(5f(x))$	

b) (4 points) Match the differential equation $\frac{d}{dt}x(t) = Ax(t)$ with the phase portraits. There is an exact match.
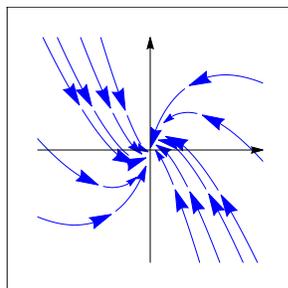
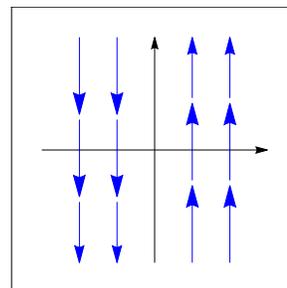| Matrix	Enter a) - d)
$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$	
$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$	
$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	
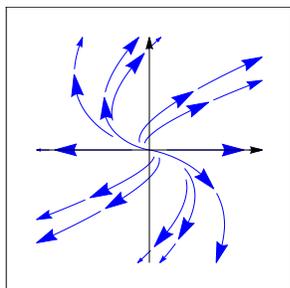
a)

b)

c)

d)

Problem 4) (10 points)

4

a) (4 points) Find all the solutions to the following system of linear equations:

$$
\begin{vmatrix}
x & + & y & + & z & + & u & + & v & + & w & = & 6 \\
x & - & y & + & z & - & u & + & v & - & w & = & 0 \\
x & + & y & + & z & + & u & - & v & - & w & = & 2
\end{vmatrix}
$$

b) (3 points) The system in a) can be written in matrix form as $A\vec{x} = \vec{b}$. Find a basis for the kernel of A.

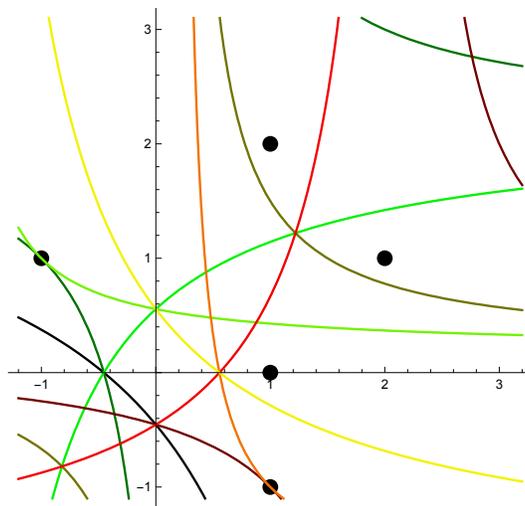c) (3 points) Find a basis for the image of A.

Problem 5) (10 points)

Using the least square method, find the hyperbola

$$xy + ax + by = 1$$

which best fits the data points (x, y):

$$\{(1, 0), (-1, 1), (1, 2), (1, -1), (2, 1)\} .$$

Problem 6) (10 points)

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

a) (3 points) Find a matrix B which is diagonal and similar to A.

b) (3 points) Find a matrix S such that $B = S^{-1}AS$ is that diagonal matrix obtained in a).
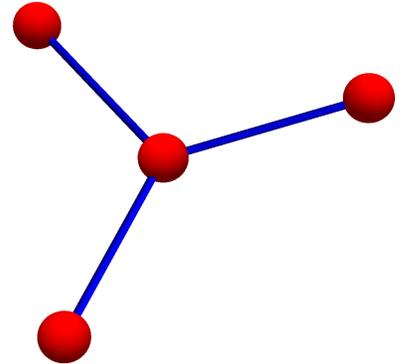
c) (2 points) Solve the discrete dynamical system $\vec{v}(t + 1) = A\vec{v}(t)$, for which the initial condition is $\vec{v}(0) = \begin{bmatrix} 11 \\ 11 \\ 2 \end{bmatrix}$.

d) (2 points) Is the system defined in c) stable?

Problem 7) (10 points)

The following matrix is called the **Laplacian of the star graph"**

$$A = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

a) (2 points) You are told that $\mathcal{B} = \{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}$ is an eigenbasis of A.

Find the eigenvalues of A.

b) (2 points) Find the characteristic polynomial of A.

c) (3 points) Write down the solution to $\vec{v}\,'(t) = A\vec{v}(t)$ with initial condition $\vec{v}(0) = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Side remark: You just have solved the "heat equation" on the star graph.

d) (3 points) Write down the solution to $\vec{v}(t+1) = A\vec{v}(t)$ with initial condition $\vec{v}(0) = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Problem 8) (10 points)

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$.

a) (2 points) Compute A^2 and A^{-1}.

b) (2 points) Find the eigenvalues of $A^T + A$.

c) (2 points) Find the QR decomposition of A.

d) (2 points) Find the characteristic polynomial $f_A(\lambda)$ of A.

e) (2 points) What are the algebraic and geometric multiplicities of the eigenvalues of A? Is A diagonalizable?

Problem 9) (10 points)

Remember the song:

> "Laplace, Row Reduce"
> Partitions, Triangular
> Eigenvectors, Eigenvalues
> Patterns you can use!"

a) (2 points) Find the determinant of the following matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 5 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

b) (2 points) Find the determinant of the following GCD matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{bmatrix}.$$

c) (2 points) Find the determinant of the following matrix

$$\begin{bmatrix} 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 4 & 4 & 4 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

d) (2 points) Find the determinant of the following matrix

$$\begin{bmatrix} 6 & 1 & 1 & 1 & 1 \\ 1 & 6 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 1 & 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix}.$$

e) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 10) (10 points)

Find the general solution to the following differential equations:

a) (2 points)
$$f'(t) + f(t) = t^2 + 1$$

b) (2 points)
$$f''(t) + 9f(t) = t^2 + 1$$

c) (3 points)
$$f''(t) + 2f'(t) + f(t) = t^2$$

d) (3 points)
$$f''(t) - f(t) = t^2 + e^t$$

Problem 11) (10 points)

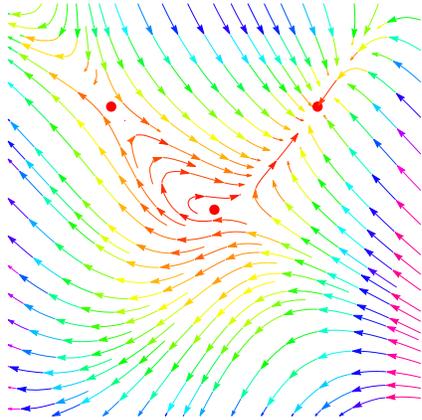We analyze the following nonlinear dynamical system

$$\frac{d}{dt}x = y - x^2$$
$$\frac{d}{dt}y = x^2 - y^2$$

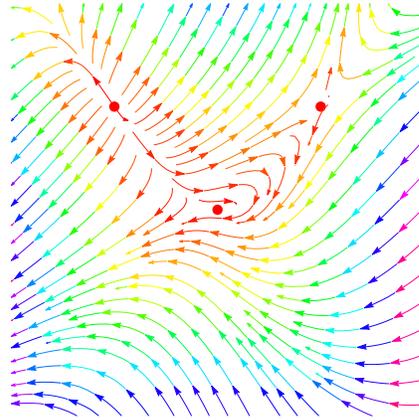a) (2 points) Find the equations of the null-clines.

b) (2 points) Find all the equilibrium points.

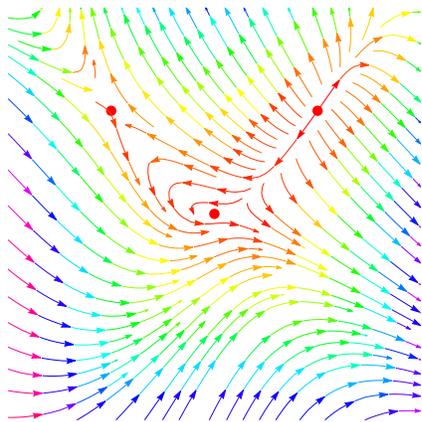c) (3 points) Analyze the stability of the equilibrium points.

d) (3 points) Which of the four phase portraits A,B,C,D below belongs to the above system? Make sure that also here, you justify your answer, as always.
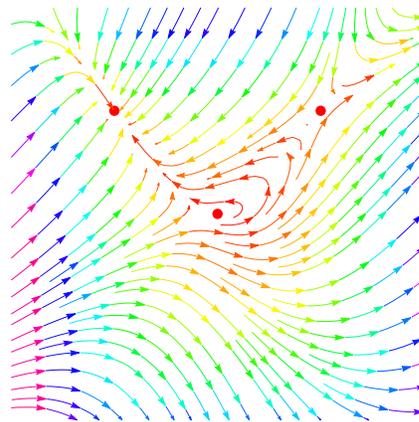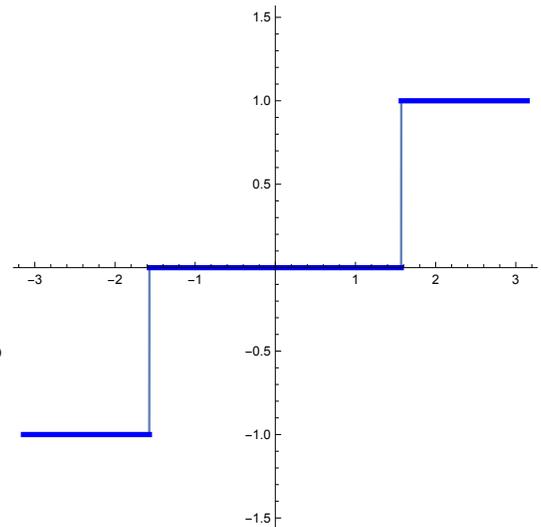

A

B

C

D

Problem 12) (10 points)

a) (7 points) Find the **Fourier series** of the function

$$f(x) = \begin{cases} 1 & \frac{\pi}{2} < x < \pi \\ 0 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1 & -\pi < x < -\frac{\pi}{2} \end{cases} .$$

The graph of the function f on $[-\pi, \pi]$ is displayed to the right.

b) (3 points) Given the Fourier coefficients a_0, a_n, b_n in the previous problem, find the sum

$$\sum_{n=0}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 .$$

Problem 13) (10 points)

a) (5 points) The partial differential equation

$$u_t = u_{xx} + u$$

is a modification of the **heat equation** on the interval $[0, \pi]$. Solve it with the initial condition

$$u(x, 0) = \sin(x) + 4\sin(7x) + 2\sin(13x) .$$

b) (5 points) The partial differential equation

$$u_{tt} = u_{xx} + u$$

is a modification of the **wave equation** on the interval $[0, \pi]$. Solve it with the initial conditions

$$u(x, 0) = 2\sin(5x), \; u_t(x, 0) = 3\sin(5x) + 6\sin(8x) .$$

P.S. As usual, we only consider sin-Fourier expansions in x.