6) Solve $f_t = 5f_{xx}$ on $[0,\pi]$ initial: $f(x,0) = |\sin(2x)|$

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-5n^2 t} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x,0)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\cdot\left(\int_0^{\pi/2} \sin(2x)\sin(nx)\,dx + \int_{\pi/2}^{\pi} -\sin(2x)\sin(nx)\,dx \right)$$

use $2\sin(nx)\sin(my) = \cos(nx-my) - \cos(nx+my)$

$$= \frac{2}{\pi}\left(\frac{1}{2}\int_0^{\pi/2} \cos((2-n)x) - \cos((2+n)x)\,dx - \frac{1}{2}\int_{\pi/2}^{\pi} \cos((2-n)x) - \cos((2+n)x)\,dx \right.$$

$$= \frac{1}{\pi}\left(\frac{1}{2-n}\sin((2-n)x)\Big|_0^{\pi/2} - \frac{1}{2+n}\sin((2+n)x)\Big|_0^{\pi/2} - \frac{1}{2-n}\sin((2-n)x)\Big|_{\pi/2}^{\pi} + \frac{1}{2+n}\sin((2+n)x)\Big|_{\pi/2}^{\pi} \right.$$

$$= \frac{1}{\pi}\left(\frac{2}{2-n}\sin\left((2-n)\cdot\frac{\pi}{2}\right) - \frac{2}{2+n}\sin\left((2+n)\pi/2\right) \right)$$

$\downarrow$ using $\sin(A+B) = \sin A\cos B + \cos A \sin B$

$$= \frac{1}{\pi}\left(\frac{2}{2-n}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{2+n}\sin\left(\frac{n\pi}{2}\right) \right) = \boxed{\frac{8\sin\left(\frac{n\pi}{2}\right)}{(4-n^2)\pi}}$$

7) $u_t = u_{xxxx} + u_{xx}$ $\quad u(0) = |\sin(2x)|$

Operator is $D^4 + D^2$ which has eigenvectors $\sin(nx)$ with eigenvalues $n^4 - n^2$

so for $f(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)$ we have the solution $\sum_{n=1}^{\infty} b_n \sin(nx) e^{(n^4-n^2)t}$

where b_n are the sin-Fourier coefficients for $|\sin(2x)|$ which are same as in question 6 over $[0,\pi]$

8) $u_{tt} = u_{xx}$ initial conditions: $u(x,0) = 0 \quad u_t(x,0) = \begin{cases} \sin(3x) & x \in [0, \pi/2] \\ 0 & x \in [\pi/2, \pi] \end{cases}$

$$u(0,t) = u(\pi,t) = 0$$

has solution: $u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(n\overset{c=1 \text{ from wave equation}}{c}t) + \frac{b_n}{nc}\sin(nx)\sin(nct)$

$a_n = 0$ since $u(x,0) = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} \sin(3x)\cdot\sin(nx)\,dx + \int_{\pi/2}^{\pi} 0\cdot\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\cdot\frac{n\cos(\frac{n\pi}{2})}{n^2-9}$$

so $\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2\cos(\frac{n\pi}{2})}{\pi c(n^2-9)}\sin(nx)\sin(nt)}$.

9) First find u_h for $u_{tt} = u_{xx}$ for odd u $\Rightarrow$ only sine solutions to wave equation

$u_h = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nt)$

$u_p(t)$ that satisfies $u_{tt} = \cos(t) + \cos(3t)$

$u_p(t) = -\cos(t) - \frac{1}{9}\cos(3t)$

at $u(x,0) = 11\sin(5x) + 23\sin(7x)$ so $a_5 = 11$, $a_7 = 23$, all others $= 0$

$$\boxed{u(x,t) = 11\sin(5x)\cos(5t) + 23\sin(7x)\cos(7t) - \cos(t) - \tfrac{1}{9}\cos(3t) + \tfrac{10}{9}}$$ to get $u(x,0)$ to match

10) a) $b_{nm} = \frac{1}{\pi^2}\Big[\int_0^{\pi}\int_0^{\pi} \sin(nx)\sin(my)\,dx\,dy + \int_{-\pi}^{0}\int_{-\pi}^{0} \sin(nx)\sin(my)\,dx\,dy$

$- \int_0^{\pi}\int_{-\pi}^{0} \sin(nx)\sin(my)\,dx\,dy - \int_{-\pi}^{0}\int_0^{\pi} \sin(nx)\sin(my)\,dx\,dy\Big]$

$$\boxed{b_{nm} = \frac{4}{\pi^2}\,\frac{(\cos(m\pi)-1)(\cos(n\pi)-1)}{m\cdot n}}$$

b) eigenvalues of $D_{xx} + D_{yy}$ for $\sin(nx)\sin(my)$ are $-n^2-m^2$ so evolves like $e^{-n^2-m^2t}\sin(my)\sin(nx)$

so $\boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx)\sin(my)e^{-n^2-m^2t}}$