Math21b Solutions Week 3 Lecture 7

3.1: [10,22],34,42,44,54,38*,48* TF8,50*

3.1.10 Solving the system $A\vec{x} = \vec{0}$ we find that $\ker(A) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

3.1.22 Compare with the solution to Exercise 21.

$$\left[\begin{array}{cc:c} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

This computation shows that the third column vector of A, $\vec{v}_3$, is a linear combination of the first two, Thus, only the first two vectors are independent, and the image is a plane in $\mathbb{R}^3$.

3.1.34 To describe a subset of $\mathbb{R}^3$ as a kernel means to describe it as an intersection of planes (think about it). By inspection, the given line is the intersection of the planes

$$\begin{aligned} x + y &= 0 \quad \text{and} \\ 2x + z &= 0. \end{aligned}$$

This means that the line is the kernel of the linear transformation $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + z \end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

3.1.42 Using the hint, we see that the vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ is in the image of A if

$$\begin{array}{cccccl} y_1 & & -3y_3 & +2y_4 & = 0 & \text{and} \\ & y_2 & -2y_3 & +y_4 & = 0. & \end{array}$$

This means that $\text{im}(A)$ is the kernel of the matrix $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$.

3.1.44 a Yes; by construction of the echelon form, the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).

b No; as a counterexample, consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, with $\text{im}(A) = \text{span}(\vec{e}_2)$, but $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, with $\text{im}(B) = \text{span}(\vec{e}_1)$.

3.1.54 a If no error occurred, then $\vec{w} = \vec{v} = M\vec{u}$, and $H\vec{w} = H(M\vec{u}) = \vec{0}$, by Exercise 53b.

If an error occurred in the ith component, then $\vec{w} = \vec{v} + \vec{e}_i = M\vec{u} + \vec{e}_i$, so that

$H\vec{w} = H(M\vec{u}) + H\vec{e}_i = i\text{th column of } H.$

Since the columns of H are all different, this method allows us to find out where an error occurred.

b $H\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ = seventh column of H: an error occurred in the seventh component of $\vec{v}$.

$$\text{Therefore } \vec{v} = \vec{w} + \vec{e}_7 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

3.1.38 a If a vector $\vec{x}$ is in $\ker(A^k)$, that is, $A^k\vec{x} = \vec{0}$, then $\vec{x}$ is also in $\ker(A^{k+1})$, since $A^{k+1}\vec{x} = AA^k\vec{x} = A\vec{0} = \vec{0}$.

Therefore, $\ker(A) \subseteq \ker(A^2) \subseteq \ker(A^3) \subseteq \dots$

Exercise 37 shows that these kernels need not be equal.

b If a vector $\vec{y}$ is in $\text{im}(A^{k+1})$, that is, $\vec{y} = A^{k+1}\vec{x}$ for some $\vec{x}$, then $\vec{y}$ is also in $\text{im}(A^k)$, since we can write $\vec{y} = A^k(A\vec{x})$. Therefore, $\text{im}(A) \supseteq \text{im}(A^2) \supseteq \text{im}(A^3) \supseteq \dots$.

Exercise 37 shows that these images need not be equal.

3.1.48 a $\vec{w} = A\vec{x}$, for some $\vec{x}$, so that $A\vec{w} = A^2\vec{x} = A\vec{x} = \vec{w}$.

b If $\text{rank}(A) = 2$, then A is invertible, and the equation $A^2 = A$ implies that $A = I_2$ (multiply by A^{-1}).

If $\text{rank}(A) = 0$ then $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Ch 3.TF.8 F; The nullity is $6 - 4 = 2$, by Theorem 3.3.7.

Ch 3.TF.50 T; Suppose $\vec{v}$ is in both $\ker(A)$ and $\text{im}(A)$, so that $\vec{v} = A\vec{w}$ for some vector $\vec{w}$. Then $\vec{0} = A\vec{v} = A^2\vec{w} = A\vec{w} = \vec{v}$, as claimed.